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TOPOLOGICAL RADICALS, IV. FRATTINI THEORY FOR BANACH LIE ALGEBRAS

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Abstract. The paper develops the theory of topological radicals of Banach Lie algebras and studies the structure of Banach Lie algebras with sufficiently many Lie subalgebras of finite codimensions – the intersection of all these subalgebras is zero. It is shown that the intersections of certain families of Lie subalgebras (closed Lie subalgebras of finite codimension, closed Lie ideals of finite codimension, closed maximal Lie subalgebras of finite codimension, closed maximal Lie ideals of finite codimension) correspond to different preradicals, and that these preradicals generate the same radical, *the Frattini radical*. The main attention is given to structural properties of Frattini-semisimple Banach Lie algebras and, in particular, to a new infinite-dimensional phenomenon associated with the strong Frattini preradical introduced in this paper. A constructive description of Frattini-free Banach Lie algebras is obtained.

1. Introduction

In this paper we pursue two interconnected aims: to develop the theory of topological radicals of Banach Lie algebras and to apply this theory to the study of the structure of Banach Lie algebras that have rich families of closed subalgebras of finite codimension.

The notion of the radical — the map that associates each Lie algebra $\mathcal{L}$ with its maximal solvable Lie ideal $\mathrm{rad}\,(\mathcal{L})$ — lies at the core of the classical theory of finite-dimensional Lie algebras. Another important map of this kind is the "nil radical" which maps a Lie algebra into its largest nilpotent ideal. In numerous other situations it is often useful and enlightening to construct specific "radical-like" maps that send Lie algebras into their Lie ideals and have some special structure properties.

The intensive study of such maps for associative algebras was extremely fruitful and produced an important branch of modern algebra — the general theory of radicals (see [Di, Sz]). A topological counterpart of this theory — the theory of topological radicals of associative normed algebras — was initiated by Dixon in [D]. He proposed a radical theory approach to the study of the existence of topologically irreducible representations of Banach algebras. Stimulated by Dixon's work, Read constructed in [R2] his famous example of a quasinilpotent operator on a Banach space that has no non-trivial closed invariant subspaces. In [ST_0, ST_1, ST_2, ST_3] the second and third authors further developed the theory of topological radicals of associative normed algebras and related this theory to many important problems in Banach algebra theory and operator theory, such as the existence of non-trivial ideals, radicality of tensor products, joint spectral radius, invariant subspaces, spectral theory of multiplication operators etc.

In this paper we introduce and study topological preradicals and radicals of Banach Lie algebras. A complex Lie algebra $\mathcal{L}$ with Lie multiplication $[\cdot,\cdot]$ is a Banach Lie algebra, if it is a Banach space in some norm $\|\cdot\|$ and there is a *multiplication constant* $t_{\mathcal{L}} > 0$ such that

$$\|[a,b]\| \leq t_{\mathcal{L}} \|a\| \|b\| \text{ for all } a,b \in \mathcal{L}.$$

For example, all Banach algebras are Banach Lie algebras with respect to the Lie multiplication $[a,b] = ab - ba$. In particular, all closed Lie subalgebras of the algebra $\mathcal{B}(X)$ of all bounded operators on a Banach space X are Banach Lie algebras. Since bilinear maps on finite-dimensional spaces are continuous, all complex finite-dimensional Lie algebras (with arbitrary norms) can be considered as Banach Lie algebras.

Denote by $\mathfrak{L}$ the class of all Banach Lie algebras. We consider the category $\overline{\mathbf{L}}$ of Banach Lie algebras with $\mathrm{Ob}\left(\overline{\mathbf{L}}\right) = \mathfrak{L}$, assuming that morphisms of $\overline{\mathbf{L}}$ are bounded homomorphisms

with dense image, and the subcategory $\mathbf{L}^{\mathrm{f}}$ of $\overline{\mathbf{L}}$ with $\mathrm{Ob}\left(\mathbf{L}^{\mathrm{f}}\right) = \mathfrak{L}^{\mathrm{f}}$ — the set of all finite-dimensional Lie algebras. It is sometimes reasonable to consider the subcategory $\mathbf{L}$ of $\overline{\mathbf{L}}$ with $\mathrm{Ob}\,(\mathbf{L}) = \mathrm{Ob}\left(\overline{\mathbf{L}}\right) = \mathfrak{L}$ and bounded epimorphisms as morphisms, but in this paper we will be mainly working in the category $\overline{\mathbf{L}}$.

A map R: $\mathfrak{L} \to \mathfrak{L}$ is a *preradical* in $\overline{\mathbf{L}}$ (in $\mathbf{L}$) if $R(\mathcal{L})$ is a closed Lie ideal of $\mathcal{L}$, for each $\mathcal{L} \in \mathfrak{L}$, and

$$f(R(\mathcal{L})) \subseteq R(\mathcal{M}) \text{ for each morphism } f\colon \mathcal{L} \longrightarrow \mathcal{M} \text{ in } \overline{\mathbf{L}} \text{ (in } \mathbf{L}).$$

The study of any preradical R leads naturally to the singling out two subclasses of $\mathfrak{L}$: the class $\mathbf{Sem}\,(R)$ of R-semisimple Lie algebras and the class $\mathbf{Rad}\,(R)$ of R-radical Lie algebras:

$$\mathbf{Sem}(R) = \{\mathcal{L} \in \mathfrak{L}\colon\ R\,(\mathcal{L}) = \{0\}\} \text{ and } \mathbf{Rad}(R) = \{\mathcal{L} \in \mathfrak{L}\colon\ R\,(\mathcal{L}) = \mathcal{L}\}.$$

A preradical R is a radical if it behaves well on ideals and quotients. In particular, $R(\mathcal{L}) \in \mathbf{Rad}(R)$ and $\mathcal{L}/R(\mathcal{L}) \in \mathbf{Sem}(R)$. Thus the radical theory approach reduces various problems concerning Lie algebras to the corresponding problems concerning separately semisimple and radical algebras. For many radicals constructed in this paper, the structure of Lie algebras in these classes is far from trivial and the study of their structure is interesting and important in many respects.

Section 2 contains some basic definitions and preliminary results of the theory of Banach Lie algebras. In Section 3 we introduce main notions of the radical theory, consider special classes of preradicals and establish some of their properties important for what follows.

Many naturally arising and important preradicals (for example, the classical nil-radical) are not radicals. It is often helpful, using some "improvement" procedures, to construct from them other preradicals with certain additional properties and, in particular, radicals associated with the initial preradicals. In Section 4 we examine these procedures. They are the Banach Lie algebraic versions of the procedures employed by Dixon for Banach associative algebras which, in turn, are counterparts of the Baer procedures for radicals of rings. They produce radicals that are either the largest out of all radicals smaller than the original preradicals, or the smallest out of all radicals larger than the original ones. We extensively use the results and constructions of this section in the further sections.

A collection $\Gamma = \{\Gamma_{\mathcal{L}}\}_{\mathcal{L}\in\mathfrak{L}}$ of families $\Gamma_{\mathcal{L}}$ of closed subspaces of Lie algebras $\mathcal{L} \in \mathfrak{L}$ is called a subspace-multifunction. Subspace-multifunctions give rise to many important preradicals on $\overline{\mathbf{L}}$. In Section 5 we study the link between subspace-multifunctions and the preradicals they generate.

In Section 6 we consider various subspace-multifunctions $\Gamma = \{\Gamma_{\mathcal{L}}\}_{\mathcal{L}\in\mathfrak{L}}$ that consist of finite-dimensional Lie subalgebras and of commutative Lie ideals of $\mathcal{L}$. We study the preradicals they generate and the corresponding radicals obtained via the methods discussed in Section 4. We show that although the preradicals generated by these subspace-multifunctions are different, the corresponding radicals often coincide and their restrictions to $\mathbf{L}^{\mathrm{f}}$ coincide with the classical radical "rad". Using ideas of Vasilescu (see [V]), we also introduce a new radical that extends "rad" to infinite dimensional Lie algebras.

Aiming to investigate in Section 8 chains of Lie subalgebras and ideals of Banach Lie algebras, we introduce and study in Section 6 the notion of a *lower finite-gap* chain of closed subspaces of a Banach space X. This means that each subspace Y in a lower finite-gap chain contains another subspace Z from this chain such that Y/Z is finite-dimensional.

In Section 7 we consider our main subject**:** the subspace-multifunctions

$$\mathfrak{S} = \{\mathfrak{S}_{\mathcal{L}}\}_{\mathcal{L}\in\mathfrak{L}} \text{ and } \mathfrak{S}^{\max} = \{\mathfrak{S}_{\mathcal{L}}^{\max}\}_{\mathcal{L}\in\mathfrak{L}},$$

where families $\mathfrak{S}_{\mathcal{L}}$ and $\mathfrak{S}_{\mathcal{L}}^{\max}$ consist, respectively, of all closed proper and closed *maximal* proper Lie subalgebras of *finite codimension* in $\mathcal{L}$; and the subspace-multifunctions

$$\mathfrak{J} = \{\mathfrak{J}_{\mathcal{L}}\}_{\mathcal{L}\in\mathfrak{L}} \text{ and } \mathfrak{J}^{\max} = \{\mathfrak{J}_{\mathcal{L}}^{\max}\}_{\mathcal{L}\in\mathfrak{L}},$$

where families $\mathfrak{J}_{\mathcal{L}}$ and $\mathfrak{J}_{\mathcal{L}}^{\max}$ consist, respectively, of all closed proper and closed *maximal* proper Lie ideals of *finite codimension* in $\mathcal{L}$. The corresponding preradicals $P_{\mathfrak{S}}$, $P_{\mathfrak{S}^{\max}}$, $P_{\mathfrak{J}}$ and $P_{\mathfrak{J}^{\max}}$

are defined by

$$P_{\mathfrak{S}}(\mathcal{L}) = \cap_{L\in\mathfrak{S}_{\mathcal{L}}} L, \quad P_{\mathfrak{S}^{\max}}(\mathcal{L}) = \cap_{L\in\mathfrak{S}^{\max}_{\mathcal{L}}} L, \quad P_{\mathfrak{J}}(\mathcal{L}) = \cap_{L\in\mathfrak{J}_{\mathcal{L}}} L \text{ and } P_{\mathfrak{J}^{\max}}(\mathcal{L}) = \cap_{L\in\mathfrak{J}^{\max}_{\mathcal{L}}} L.$$

Recall that, for finite-dimensional Lie algebras $\mathcal{L}$, $P_{\mathfrak{S}^{\max}}(\mathcal{L})$ is the Frattini ideal of $\mathcal{L}$ and $P_{\mathfrak{J}^{\max}}(\mathcal{L})$ is the Jacobson ideal of $\mathcal{L}$. The study of the above preradicals is based on the main result of [KST2] which states that if $\mathcal{L}_0$ is a maximal closed Lie subalgebra of finite codimension in a Banach Lie algebra $\mathcal{L}$, then $\mathcal{L}_0$ contains a closed Lie ideal of finite codimension. Using it, we prove that the radicals generated by the preradicals $P_{\mathfrak{S}}$, $P_{\mathfrak{S}^{\max}}$, $P_{\mathfrak{J}}$ and $P_{\mathfrak{J}^{\max}}$ coincide. The obtained radical is denoted by $\mathcal{F}$ and called *the Frattini radical*. We show that the classes of the radical Lie algebras corresponding to these preradicals and to the Frattini radical $\mathcal{F}$ coincide, while the classes of their semisimple Lie algebras satisfy the inclusions

$$\mathbf{Sem}(P_{\mathfrak{J}^{\max}}) \subset \mathbf{Sem}(P_{\mathfrak{S}^{\max}}) \subset \mathbf{Sem}(P_{\mathfrak{J}}) \subset \mathbf{Sem}(P_{\mathfrak{S}}) \subset \mathbf{Sem}(\mathcal{F})$$

and all these inclusions are proper.

In Section 8 we establish that each Banach Lie algebra $\mathcal{L} \in \mathbf{Sem}(P_{\mathfrak{J}})$ has a maximal lower finite-gap chain of closed Lie ideals between $\{0\}$ and $\mathcal{L}$. We characterize $\mathcal{F}$-semisimple Lie algebras in terms of lower finite-gap chains of Lie subalgebras: a Banach Lie algebra $\mathcal{L}$ is $\mathcal{F}$-semisimple if and only if it has a lower finite-gap chain of closed Lie subalgebras between $\{0\}$ and $\mathcal{L}$.

Making use of lower finite-gap chains of Lie ideals in Banach Lie algebras, we define another important preradical on $\overline{\mathbf{L}}$ — the strong Frattini preradical $\mathcal{F}_s$. We show that $\mathcal{F}_s(\mathcal{F}_s(\mathcal{L})) = \mathcal{F}(\mathcal{L})$ and that $\mathcal{F}_s(\mathcal{L})/\mathcal{F}(\mathcal{L})$ is commutative for each Banach Lie algebra. A Banach Lie algebra $\mathcal{L}$ is $\mathcal{F}_s$-semisimple if and only if it has a lower finite-gap chain of closed Lie ideals between $\{0\}$ and $\mathcal{L}$. Moreover, each closed Lie subalgebra of a $\mathcal{F}_s$-semisimple Lie algebra is also $\mathcal{F}_s$-semisimple.

Section 9 is devoted to the study of Frattini-free Banach Lie algebras — the Lie algebras satisfying the condition

$$P_{\mathfrak{S}^{\max}}(\mathcal{L}) = \cap_{L\in\mathfrak{S}^{\max}_{\mathcal{L}}} L = \{0\}, \text{ that is, } \mathcal{L} \in \mathbf{Sem}(P_{\mathfrak{S}^{\max}}).$$

In [K] the first author considered Frattini-free Banach Lie algebras all of whose maximal Lie subalgebras have codimension 1. In this paper we consider the general case and prove that each Frattini-free Banach Lie algebra has the largest closed solvable Lie ideal S and that this ideal has solvability index 2, that is, $[S, S]$ is commutative. We also obtain a structural description of Frattini-free Lie algebras as subdirect products of families of finite-dimensional *subsimple* Lie algebras (see Definitions 9.1 and 9.8).

At the end of the paper we consider finite-dimensional Lie algebras $\mathcal{L}$. We show that our characterization of Frattini-free Banach Lie algebras implies a transparent description of each finite-dimensional Frattini-free algebra as the direct sum of at most three summands — a semisimple Lie algebra, a commutative algebra and a semidirect product $L \oplus^{\mathrm{id}} X$, where L is a decomposable Lie algebra of operators on a finite-dimensional linear space X. This immediately gives us the description of finite-dimensional Frattini-free Lie algebras obtained by Stitzinger [S] and Towers [T]. Furthermore, using results of Marshall (see [M]) about the relation between the nil-radical of $\mathcal{L}$ and the Frattini and Jacobson ideals of $\mathcal{L}$, we obtain some inequalities that relate the Frattini and Jacobson indices of $\mathcal{L}$ to the solvability index of the nil-radical of $\mathcal{L}$.

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2. Characteristic Lie ideals and subideals of Banach Lie algebras

Let $\mathcal{L}$ be a Banach Lie algebra. A subspace L of $\mathcal{L}$ is a *Lie subalgebra* (*ideal*) if $[a, b] \in L$, for each $a, b \in L$ (respectively, $a \in L$, $b \in \mathcal{L}$). A linear map δ on $\mathcal{L}$ is a *Lie derivation* if

$$\delta([a, b]) = [\delta(a), b] + [a, \delta(b)] \text{ for } a, b \in \mathcal{L}. \tag{2.1}$$

Each $a \in \mathcal{L}$ defines a bounded Lie derivation $\mathrm{ad}\,(a)$ on $\mathcal{L}$: $\mathrm{ad}\,(a)\,x = [a, x]$.

Denote by $\mathfrak{D}\,(\mathcal{L})$ the set of all bounded Lie derivations on $\mathcal{L}$. It is a closed Lie subalgebra of the algebra $\mathcal{B}\,(\mathcal{L})$ of all bounded operators on $\mathcal{L}$ and $\mathrm{ad}(\mathcal{L}) = \{\mathrm{ad}\,(a) : a \in \mathcal{L}\}$ is a Lie ideal of

$\mathfrak{D}(\mathcal{L})$, as $[\delta,\mathrm{ad}(a)] = \mathrm{ad}(\delta(a))$. If J is a Lie ideal of $\mathcal{L}$, we denote by $\mathrm{ad}(a)|_J$ the restriction of $\mathrm{ad}(a)$ to J, and $\mathrm{ad}(\mathcal{L})|_J = \{\mathrm{ad}(a)|_J\colon a \in \mathcal{L}\}$.

A Lie ideal of $\mathcal{L}$ is called *characteristic* if it is invariant for all $\delta \in \mathfrak{D}(\mathcal{L})$.

Notation 2.1. We write $J \lhd \mathcal{L}$ if J is a closed Lie ideal of a Banach Lie algebra $\mathcal{L}$, and $J \lhd^{\mathrm{ch}} \mathcal{L}$ if J is a characteristic closed Lie ideal of $\mathcal{L}$.

It is easy to check that the centre of $\mathcal{L}$ is a characteristic Lie ideal. If $\mathcal{L}$ is commutative then $\{0\}$ and $\mathcal{L}$ are the only characteristic Lie ideals of $\mathcal{L}$. Indeed, each closed subspace of $\mathcal{L}$ is a Lie ideal, each bounded operator on $\mathcal{L}$ is a derivation and only $\{0\}$ and $\mathcal{L}$ are invariant for $\mathcal{B}(\mathcal{L})$.

The following lemma shows that subspaces of $\mathcal{L}$ invariant for all bounded Lie isomorphisms are characteristic ideals.

Lemma 2.2. *Let J be a closed linear subspace of a Banach Lie algebra $\mathcal{L}$ invariant for all bounded Lie isomorphisms of $\mathcal{L}$. Then $J \lhd^{\mathrm{ch}} \mathcal{L}$.*

Proof. For each $\delta \in \mathfrak{D}(\mathcal{L})$, $\exp(t\delta) = \sum_{i=0}^{\infty} \frac{t^n\delta^n}{n!}$, $t \in \mathbb{R}$, is a one-parameter group of bounded Lie automorphisms of $\mathcal{L}$: $\exp(t\delta)([a,b]) = [\exp(t\delta)(a), \exp(t\delta)(b)]$ for all $a, b \in \mathcal{L}$. Hence $\exp(t\delta)(J) \subseteq J$. Since $\delta(a) = \lim_{t\to 0}(\exp(t\delta)(a) - a)/t$, for each $a \in \mathcal{L}$, J is invariant for δ, so it is a characteristic Lie ideal of $\mathcal{L}$. $\square$

Clearly, the intersection and the closed linear span of a family of characteristic Lie ideals are characteristic Lie ideals.

Lemma 2.3. *Let $\mathcal{L}$ be a Banach Lie algebra, let $J \lhd^{\mathrm{ch}} \mathcal{L}$ and $q : \mathcal{L} \longrightarrow \mathcal{L}/J$ be the quotient map. If $I \lhd^{\mathrm{ch}} \mathcal{L}/J$ then $q^{-1}(I) \lhd^{\mathrm{ch}} \mathcal{L}$.*

Proof. As J is a characteristic Lie ideal, $\delta(J) \subseteq J$, for each $\delta \in \mathfrak{D}(\mathcal{L})$. Hence the quotient map $\delta^q\colon q(x) \to q(\delta(x))$ on $\mathcal{L}/J$ is, clearly, a derivation of $\mathcal{L}/J$. Since $I \lhd^{\mathrm{ch}} \mathcal{L}/J$, we have $\delta^q(I) \subseteq I$. This means that $\delta(q^{-1}(I)) \subseteq q^{-1}(I)$, so that $q^{-1}(I)$ is a characteristic Lie ideal of $\mathcal{L}$. $\square$

If $I \lhd J \lhd \mathcal{L}$ then I is not necessarily a Lie ideal of $\mathcal{L}$. For example, each subspace I of a commutative ideal J of a Lie algebra $\mathcal{L}$ is not necessarily a Lie ideal of $\mathcal{L}$ (e.g. subspaces of a Banach space X in the semidirect product $\mathcal{L} = \mathcal{B}(X) \oplus^{\mathrm{id}} X$ (see (3.10)) are not Lie ideals of $\mathcal{L}$).

Statements (i) and (ii) in the following lemma are related to Lemma 0.4 [St], and (iii) belongs to the mathematical folklore; for the sake of completeness we present the proofs.

Lemma 2.4. (i) *If $I \lhd^{\mathrm{ch}} J \lhd \mathcal{L}$ then $I \lhd \mathcal{L}$.*

(ii) *If $I \lhd^{\mathrm{ch}} J \lhd^{\mathrm{ch}} \mathcal{L}$ then $I \lhd^{\mathrm{ch}} \mathcal{L}$.*

(iii) *If $J \lhd \mathcal{L}$ and $J = \overline{[J,J]}$ then $J \lhd^{\mathrm{ch}} \mathcal{L}$.*

Proof. (i) As $J \lhd \mathcal{L}$, $\mathrm{ad}(\mathcal{L})|_J$ is a Lie subalgebra of $\mathfrak{D}(J)$. Hence I is invariant for $\mathrm{ad}(\mathcal{L})|_J$. Thus I is a Lie ideal of $\mathcal{L}$.

(ii) We have $\delta(J) \subseteq J$ and $\delta|_J \in \mathfrak{D}(J)$, for all $\delta \in \mathfrak{D}(\mathcal{L})$, and $\Delta(I) \subseteq I$ for all $\Delta \in \mathfrak{D}(J)$. Hence $\delta(I) \subseteq I$ for all $\delta \in \mathfrak{D}(\mathcal{L})$. Thus I is a characteristic Lie ideal of $\mathcal{L}$.

(iii) By (2.1), for each $\delta \in \mathfrak{D}(\mathcal{L})$, we have $\delta([J,J]) \subseteq [\delta(J),J] + [J,\delta(J)] \subseteq [J,\mathcal{L}] \subseteq J$. As δ is bounded, $\delta(J) = \delta(\overline{[J,J]}) \subseteq J$. Hence J is a characteristic Lie ideal of $\mathcal{L}$. $\square$

The existence of Lie ideals and characteristic Lie ideals of finite codimension was studied in [KST1, KST2]. We will often use the following result obtained in [KST2].

Theorem 2.5. [KST2] *Let a Banach Lie algebra $\mathcal{L}$ have a closed proper Lie subalgebra $\mathcal{L}_0$ of finite codimension. Then $\mathcal{L}$ has a closed proper Lie ideal of finite codimension. In addition,*

(i) *If $\mathcal{L}_0$ is maximal, then $\mathcal{L}_0$ contains a closed Lie ideal of $\mathcal{L}$ of finite codimension.*

(ii) *If $\mathcal{L}$ is non-commutative, it has a proper closed characteristic Lie ideal of finite codimension.*

Corollary 2.6. *Let $\mathcal{L}$ be a Banach Lie algebra and J be a non-commutative infinite-dimensional closed Lie ideal of $\mathcal{L}$. If J has a proper closed Lie subalgebra of finite codimension, then J contains a closed Lie ideal I of $\mathcal{L}$ that has non-zero finite codimension in J.*

If, in addition, J is a characteristic Lie ideal of $\mathcal{L}$, then I is also a characteristic Lie ideal.

Proof. By Theorem 2.5(ii), J has a proper closed characteristic Lie ideal I of finite codimension. By Lemma 2.4, I is a Lie ideal of $\mathcal{L}$; if J is characteristic then I is also characteristic. $\square$

Definition 2.7. *A Lie subalgebra I of a Banach Lie algebra $\mathcal{L}$ is called a* Lie subideal *(more precisely n-*subideal*), if there are Lie subalgebras $J_1,..., J_n$ of $\mathcal{L}$ such that $J_0 := I \subseteq J_1 \subseteq \cdots \subseteq J_n = \mathcal{L}$ and each J_i is a Lie ideal of J_{i+1}. We write $I \lll \mathcal{L}$ if I is closed. In this case all J_i can be assumed to be closed (otherwise one can replace them by their closures).*

In some important cases Lie subideals are automatically ideals. Recall that a finite-dimensional Lie algebra is *semisimple*, if it has no non-zero commutative Lie ideals.

Lemma 2.8. *Let $L \lll \mathcal{L}$. If L is a finite-dimensional semisimple Lie algebra, then it is a Lie ideal of $\mathcal{L}$.*

Proof. Let $L = J_0 \lhd J_1 \lhd \cdots \lhd J_n = \mathcal{L}$. Since L is semisimple, it is well known that $[L, L] = L$. Hence, by Lemma 2.4(iii), $L \lhd^{\mathrm{ch}} J_1$. Therefore, by Lemma 2.4(i), L is a Lie ideal of J_2. Repeating the argument, we obtain that L is a Lie ideal of $\mathcal{L}$. $\square$

Corollary 2.9. *Each Lie subideal of a finite-dimensional semisimple Lie algebra is a Lie ideal.*

Proof. Let $L = J_0 \lhd J_1 \lhd \cdots \lhd J_n = \mathcal{L}$. As $\mathcal{L}$ is semisimple, each Lie ideal of $\mathcal{L}$ is a semisimple Lie algebra. Hence L is semisimple. By Lemma 2.8, it is a Lie ideal of $\mathcal{L}$. $\square$

3. Preradicals

3.1. **Basic properties.** Recall that $\mathfrak{L}$ denotes the class of all Banach Lie algebras and that the symbol $J \lhd^{\mathrm{ch}} \mathcal{L}$ means that J is a closed characteristic ideal of $\mathcal{L}$.

Now we will define a notion which plays the central role in this paper.

Definition 3.1. *A map R on $\mathfrak{L}$ that sends each $\mathcal{L} \in \mathfrak{L}$ into a closed Lie ideal $R(\mathcal{L})$ of $\mathcal{L}$ is a topological preradical in* $\mathbf{L}$ *(in* $\overline{\mathbf{L}}$*) if*

$$f(R(\mathcal{L})) \subseteq R(\mathcal{M}) \text{ for each morphism } f\colon \mathcal{L} \longrightarrow \mathcal{M} \text{ in } \mathbf{L} \ (\text{in } \overline{\mathbf{L}}). \tag{3.1}$$

Remark 3.2. We will omit the word "topological" in all notions of the radical theory, because we do not consider here the radical theory in the purely algebraic setting.

For example, the map R: $\mathcal{L} \mapsto \overline{[\mathcal{L}, \mathcal{L}]}$, for all $\mathcal{L} \in \mathfrak{L}$, is a preradical.

If R is a preradical then it follows from (3.1) that

$$\text{if } f : \mathcal{L} \longrightarrow \mathcal{M} \text{ is a bounded Lie isomorphism, then so is } f : \ R(\mathcal{L}) \longrightarrow R(\mathcal{M}). \tag{3.2}$$

Corollary 3.3. *Let $I \lhd \mathcal{L} \in \mathfrak{L}$ and $q : \mathcal{L} \longrightarrow \mathcal{L}/I$ be the quotient map. For each preradical R,*

(i) $R(\mathcal{L}) \lhd^{\mathrm{ch}} \mathcal{L}$.
(ii) $R(I) \lhd \mathcal{L}$ *and* $q^{-1}(R(\mathcal{L}/I)) \lhd \mathcal{L}$.
(iii) *If* $I \lhd^{\mathrm{ch}} \mathcal{L}$ *then* $R(I) \lhd^{\mathrm{ch}} \mathcal{L}$ *and* $q^{-1}(R(\mathcal{L}/I)) \lhd^{\mathrm{ch}} \mathcal{L}$.

Proof. Part (i) follows from Lemma 2.2 and (3.2).

(ii) By (i), $R(I)$ is a characteristic Lie ideal of I. Hence, by Lemma 2.4(i), $R(I) \lhd \mathcal{L}$. As $R(\mathcal{L}/I) \lhd \mathcal{L}/I$, we have that $q^{-1}(R(\mathcal{L}/I)) \lhd \mathcal{L}$.

(iii) Let $I \lhd^{\mathrm{ch}} \mathcal{L}$. Then, by (i), $R(I) \lhd^{\mathrm{ch}} I$. Hence, by Lemma 2.4(ii), $R(I) \lhd^{\mathrm{ch}} \mathcal{L}$.

By (i), $R(\mathcal{L}/I) \lhd^{\mathrm{ch}} \mathcal{L}/I$. Hence, by Lemma 2.3, $q^{-1}(R(\mathcal{L}/I)) \lhd^{\mathrm{ch}} \mathcal{L}$. $\square$

We are interested in preradicals with some additional algebraic properties: R is called

$$\textit{lower stable} \text{ if } R(R(\mathcal{L})) = R(\mathcal{L}) \text{ for all } \mathcal{L} \in \mathfrak{L}; \tag{3.3}$$

$$\textit{upper stable} \text{ if } R(\mathcal{L}/R(\mathcal{L})) = \{0\} \text{ for all } \mathcal{L} \in \mathfrak{L}; \tag{3.4}$$

$$\textit{balanced} \text{ if } R(I) \subseteq R(\mathcal{L}) \text{ for all } I \lhd \mathcal{L} \in \mathfrak{L}; \tag{3.5}$$

$$\textit{hereditary} \text{ if } R(I) = I \cap R(\mathcal{L}) \text{ for all } I \lhd \mathcal{L} \in \mathfrak{L}. \tag{3.6}$$

Definition 3.4. *A preradical is called*

(i) *an* under radical *if it is lower stable and balanced.*
(ii) *an* over radical *if it is upper stable and balanced.*
(iii) aradical *if it is lower stable, upper stable and balanced.*

For example, the maps R_0: $\mathcal{L} \mapsto \{0\}$ and R_1: $\mathcal{L} \mapsto \mathcal{L}$, for all $\mathcal{L} \in \mathfrak{L}$, are radicals.

Remark 3.5. The statement "$I \lhd \mathcal{L}$ implies $R(I) \lhd \mathcal{L}$" proved in Corollary 3.3(ii) is not generally true for associative algebras, so it was included as a separate condition in the definition of the topological radical in [D].

Let R be a preradical. A Banach Lie algebra $\mathcal{L}$ is called

$$1)\ \textit{R-semisimple} \text{ if } R(\mathcal{L}) = \{0\}, \qquad 2)\ \textit{R-radical} \text{ if } R(\mathcal{L}) = \mathcal{L}. \tag{3.7}$$

Set $\mathbf{Sem}(R) = \{\mathcal{L} \in \mathfrak{L}\colon R(\mathcal{L}) = \{0\}\}$ and $\mathbf{Rad}(R) = \{\mathcal{L} \in \mathfrak{L}\colon R(\mathcal{L}) = \mathcal{L}\}$.

Lemma 3.6. *Let R be a preradical, let $I \lhd \mathcal{L}$ and let $q : \mathcal{L} \longrightarrow \mathcal{L}/I$ be the quotient map.*

(i) *If $\mathcal{L} \in \mathbf{Rad}(R)$ then $q(\mathcal{L}) \in \mathbf{Rad}(R)$.*
(ii) *If $q(\mathcal{L}) \in \mathbf{Sem}(R)$ then $R(\mathcal{L}) \subseteq I$.*
(iii) *Let R be balanced. If $\mathcal{L} \in \mathbf{Sem}(R)$ then $I \in \mathbf{Sem}(R)$.*
(iv) *Let R be balanced and upper stable. If I and $q(\mathcal{L})$ belong to $\mathbf{Rad}(R)$ then $\mathcal{L} \in \mathbf{Rad}(R)$.*
(v) *Let R be balanced and lower stable. If I and $q(\mathcal{L})$ belong to $\mathbf{Sem}(R)$ then $\mathcal{L} \in \mathbf{Sem}(R)$.*

Proof. (i) As $R(\mathcal{L}) = \mathcal{L}$, we have $q(\mathcal{L}) = q(R(\mathcal{L})) \overset{(3.1)}{\subseteq} R(q(\mathcal{L})) \subseteq q(\mathcal{L})$. Hence $q(\mathcal{L}) = R(q(\mathcal{L}))$.

(ii) We have $q(R(\mathcal{L})) \overset{(3.1)}{\subseteq} R(q(\mathcal{L})) = \{0\}$. Hence $R(\mathcal{L}) \subseteq I$.

(iii) If $I \lhd \mathcal{L}$ then $R(I) \subseteq R(\mathcal{L}) = \{0\}$.

(iv) As R is balanced and $I \in \mathbf{Rad}(R)$, we have $I = R(I) \subseteq R(\mathcal{L})$. Hence there is a quotient map p: $\mathcal{L}/I \to \mathcal{L}/R(\mathcal{L})$. As R is upper stable and $\mathcal{L}/I \in \mathbf{Rad}(R)$,

$$\mathcal{L}/R(\mathcal{L}) = p(\mathcal{L}/I) = p(R(\mathcal{L}/I)) \subseteq R(p(\mathcal{L}/I)) = R(\mathcal{L}/R(L)) = \{0\}.$$

Thus $\mathcal{L} = R(\mathcal{L})$.

(v) It follows from (ii) that $R(\mathcal{L}) \subseteq I$. Then $R(\mathcal{L}) \lhd I$. As R is balanced, $R(R(\mathcal{L})) \subseteq R(I) = \{0\}$. As R is lower stable, $R(\mathcal{L}) = R(R(\mathcal{L})) = \{0\}$. □

In particular, it follows from Lemma 3.6 that if R is a radical then both classes $\mathbf{Sem}(R)$ and $\mathbf{Rad}(R)$ are closed under extensions.

There is a natural order in the class of all preradicals. If R and T are preradicals, we write

$$T \le R, \text{ if } T(\mathcal{L}) \subseteq R(\mathcal{L}) \text{ for all } \mathcal{L} \in \mathfrak{L}. \tag{3.8}$$

We write $T < R$, if $T \le R$ and there is a Banach Lie algebra $\mathcal{L}$ such that $T(\mathcal{L}) \ne R(\mathcal{L})$.

If $T \le R$ then $\mathbf{Sem}(R) \subseteq \mathbf{Sem}(T)$ and $\mathbf{Rad}(T) \subseteq \mathbf{Rad}(R)$. Conversely, the following result shows that in many cases the order is determined by these inclusions.

Proposition 3.7. *Let T, R be preradicals.*

(i) *If T is lower stable and R is balanced then $\mathbf{Rad}(T) \subseteq \mathbf{Rad}(R)$ implies $T \le R$.*
(ii) *If T and R are under radicals then $\mathbf{Rad}(T) = \mathbf{Rad}(R)$ if and only if $T = R$.*
(iii) *If R is upper stable then $\mathbf{Sem}(R) \subseteq \mathbf{Sem}(T)$ implies $T \le R$.*
(iv) *If T and R are upper stable then $\mathbf{Sem}(T) = \mathbf{Sem}(R)$ if and only if $T = R$.*

(v) *Let $T \le R$, T be balanced and $I \lhd \mathcal{L}$. If $T(I) = I$ and $R(\mathcal{L}/I) = \{0\}$ then $T(\mathcal{L}) = R(\mathcal{L}) = I$.*

Proof. (i) As T is lower stable, $T(\mathcal{L}) \in \mathbf{Rad}\,(T)$ for each $\mathcal{L} \in \mathfrak{L}$. Hence $T(\mathcal{L}) \in \mathbf{Rad}\,(R)$. Then $T(\mathcal{L}) = R(T(\mathcal{L}))$. Since R is balanced and $T(\mathcal{L}) \lhd \mathcal{L}$, we have $T(\mathcal{L}) = R(T(\mathcal{L})) \subseteq R(\mathcal{L})$.

(iii) As R is upper stable, $\mathcal{L}/R(\mathcal{L}) \in \mathbf{Sem}\,(R)$ for each $\mathcal{L} \in \mathfrak{L}$. Hence $\mathcal{L}/R(\mathcal{L}) \in \mathbf{Sem}\,(T)$. By Lemma 3.6(ii), $T(\mathcal{L}) \subseteq R(\mathcal{L})$. Part (iii) is proved.

Part (ii) follows from (i), and (iv) from (iii).

(v) As $R(\mathcal{L}/I) = \{0\}$, we have from Lemma 3.6(ii) that $R(\mathcal{L}) \subseteq I$. As T is balanced,

$$I = T(I) \subseteq T(\mathcal{L}) \subseteq R(\mathcal{L}) \subseteq I.$$

□

Corollary 3.8. (i) *If R is a radical then $R(\mathcal{L}) \in \mathbf{Rad}\,(R)$ and $\mathcal{L}/R(\mathcal{L}) \in \mathbf{Sem}\,(R)$ for each $\mathcal{L} \in \mathfrak{L}$. Moreover, $R(\mathcal{L})$ contains each R-radical Lie ideal of $\mathcal{L}$.*

(ii) *Let T and R be radicals. Then*

$$T = R \iff \mathbf{Rad}\,(T) = \mathbf{Rad}\,(R) \iff \mathbf{Sem}\,(T) = \mathbf{Sem}\,(R).$$

Proof. We only need to prove that $R(\mathcal{L})$ contains each R-radical Lie ideal I of $\mathcal{L}$. Indeed, as R is balanced, $I = R(I) \subseteq R(\mathcal{L})$. □

Definition 3.9. *Let R be a preradical. A closed Lie ideal I of a Banach Lie algebra $\mathcal{L}$ is called R-primitive if $\mathcal{L}/I$ is R-semisimple.* $\mathrm{Prim}_R(\mathcal{L})$ *denotes the set of all R-primitive ideals of $\mathcal{L}$.*

The following useful result was proved in [ST$_1$, Theorem 2.11] for radicals in normed associative algebras. We will just check that the proof also works for Banach Lie algebras.

Theorem 3.10. *Let R be a preradical and $\mathcal{L}$ be a Banach Lie algebra. Then*
(i) *the intersection of any family of R-primitive Lie ideals of $\mathcal{L}$ is R-primitive*;
(ii) *each R-primitive Lie ideal of $\mathcal{L}$ contains $R(\mathcal{L})$*;
(iii) *if R is an upper stable then $R(\mathcal{L})$ is the smallest R-primitive ideal of $\mathcal{L}$.*

Proof. (i) Let $\{J_\lambda\}$ be a family of R-primitive ideals of $\mathcal{L}$ and $J = \cap J_\lambda$. Since $J \subseteq J_\lambda$, there is a bounded epimorphism $p_\lambda : \mathcal{L}/J \longrightarrow \mathcal{L}/J_\lambda$ with $q_\lambda = p_\lambda q$, where $q_\lambda : \mathcal{L} \longrightarrow \mathcal{L}/J_\lambda$ and $q : \mathcal{L} \longrightarrow \mathcal{L}/J$ are quotient maps. Therefore $p_\lambda(R(\mathcal{L}/J)) \overset{(3.1)}{\subseteq} R(\mathcal{L}/J_\lambda) = \{0\}$, so that $R(\mathcal{L}/J) \subseteq J_\lambda/J$ for every λ. Then $q^{-1}(R(\mathcal{L}/J)) \subseteq \cap J_\lambda = J$, whence $R(\mathcal{L}/J) = \{0\}$.

Part (ii) follows from Lemma 3.6(ii).

(iii) If R is upper stable then, by (3.4), $R(\mathcal{L}) \in \mathrm{Prim}_R(\mathcal{L})$. So $R(\mathcal{L})$ is the smallest R-primitive ideal of $\mathcal{L}$. □

Note that in general not every ideal containing $R(\mathcal{L})$ is R-primitive.

3.2. **Preradicals of direct and semidirect products.** Many examples below will be based on the following well known construction (see [Bo, Sec 1.8]).

Let L_1, L_0 be Banach Lie algebras and φ be a bounded Lie homomorphism from L_1 into $\mathfrak{D}(L_0)$. Endowing their direct Banach space sum $L_1 \dotplus L_0$ with Lie multiplication given by

$$[(a;x),(b;y)] = ([a,b]; \varphi(a)\,y - \varphi(b)\,x + [x,y]), \text{ for } a,b \in L_1,\ x,y \in L_0, \tag{3.9}$$

we get the *semidirect product* $\mathcal{L} = L_1 \oplus^{\varphi} L_0$. It is a Lie algebra. Moreover, it is a Banach Lie algebra with norm $\|(a;x)\| = \max\{\|a\|, \|x\|\}$ and the multiplication constant $t_{\mathcal{L}} = \max\{t_{L_1}, 2\|\varphi\| + t_{L_0}\}$. Identify $\{0\} \oplus^{\varphi} L_0$ and L_0. Then $L_0 \lhd \mathcal{L}$ and $\mathcal{L}/L_0$ is isomorphic to L_1.

If $\varphi = 0$, we obtain the direct product $L_1 \oplus L_0$.

If L_1 is a Lie subalgebra of $\mathcal{B}(L_0)$ then we take $\varphi = \mathrm{id}$ and write $L_1 \oplus^{\mathrm{id}} L_0$.

Let L_0 be commutative. Denote $X = L_0$. Then X is a Banach space and $\mathfrak{D}(X) = \mathcal{B}(X)$, as (2.1) holds for all $x, y \in X$ and $T \in \mathcal{B}(X)$. Let us identify L_1 with the Lie subalgebra $\varphi(L_1)$ of

$\mathcal{B}(X)$ and write ax instead of $\varphi(a)x$, for $a \in L_1$ and $x \in X$. Then the above construction gives us the semidirect product $\mathcal{L} = L_1 \oplus^{\mathrm{id}} X$ with binary operation

$$[(a;x),(b;y)] = ([a,b]; ay - bx) \text{ for } a,b \in L_1 \text{ and } x,y \in X. \tag{3.10}$$

Let M be a closed Lie subalgebra of L_1 and Y be a closed subspace of X invariant for all operators in M. Then $M \oplus^{\mathrm{id}} Y$ can be identified with the closed subalgebra of $\mathcal{L}$ consisting of all pairs $(a;x)$ with $a \in M$ and $x \in Y$.

Consider now the behavior of the semidirect product with respect to preradicals.

Proposition 3.11. *Let $\mathcal{L} = L_1 \oplus^{\varphi} L_0$ and let R be a preradical. Then*

(i) $R(\mathcal{L}) \subseteq R(L_1) \oplus^{\varphi} L_0$.

(ii) *Let R be balanced. Then $R(R(L_1) \oplus^{\varphi} L_0) \subseteq R(\mathcal{L})$ and*

1) *if $\varphi = 0$, so that $\mathcal{L} = L_1 \oplus L_0$, then $R(\mathcal{L}) = R(L_1) \oplus R(L_0)$.*

2) *if R is upper stable and $L_0, L_1 \in \mathbf{Rad}(R)$, then $\mathcal{L} \in \mathbf{Rad}(R)$.*

3) *if $L_1 \in \mathbf{Sem}(R)$ then $R^2(\mathcal{L}) \subseteq R(L_0) \subseteq R(\mathcal{L}) \subseteq L_0$. If R is also lower stable then $R(\mathcal{L}) = R(L_0)$.*

Proof. (i) The map $f\colon \mathcal{L} \longrightarrow L_1$ defined by $f((a;x)) = a$, for all $(a;x) \in \mathcal{L}$, is a homomorphism from $\mathcal{L}$ onto L_1. As R is a preradical, $f(R(\mathcal{L})) \subseteq R(L_1)$. Thus $R(\mathcal{L}) \subseteq R(L_1) \oplus^{\varphi} L_0$.

(ii) Let R be balanced. As $R(L_1) \oplus^{\varphi} L_0$ is a closed Lie ideal of $\mathcal{L}$, $R(R(L_1) \oplus^{\varphi} L_0) \subseteq R(\mathcal{L})$.

1) If $\varphi = 0$ then, by (i), $R(\mathcal{L}) \subseteq R(L_1) \oplus L_0$ and $R(\mathcal{L}) \subseteq L_1 \oplus R(L_0)$. Hence $R(\mathcal{L}) \subseteq R(L_1) \oplus R(L_0)$. As L_1 and L_0 are closed Lie ideals of $\mathcal{L}$, we have $R(L_1) \subseteq R(\mathcal{L})$ and $R(L_0) \subseteq R(\mathcal{L})$. Hence $R(\mathcal{L}) = R(L_1) \oplus R(L_0)$.

Part 2) follows from Lemma 3.6(iv).

3) As R is balanced and $R(L_1) = 0$, (i) implies $R(L_0) \subseteq R(\mathcal{L}) \subseteq L_0$. As $R(\mathcal{L}) \lhd L_0$ and R is balanced, we have $R^2(\mathcal{L}) \subseteq R(L_0)$. If, in addition, R is lower stable then $R^2(\mathcal{L}) = R(\mathcal{L})$ implies $R(\mathcal{L}) = R(L_0)$. $\square$

In particular, if R is a radical then a semidirect product of R-radical algebras is R-radical.

We will define now the direct product of an arbitrary family of Banach Lie algebras.

Definition 3.12. *Let $\{\mathcal{L}_\lambda\}_{\lambda\in\Lambda}$ be Banach Lie algebras with the multiplication constants t_λ satisfying $t_\Lambda = \sup\{t_\lambda\} < \infty$. The Banach Lie algebra*

$$\mathcal{L} = \oplus_\Lambda \mathcal{L}_\lambda = \{a = (a_\lambda)_{\lambda\in\Lambda} \colon a_\lambda \in \mathcal{L}_\lambda \text{ and } \|a\| = \sup\{\|a_\lambda\|_{\mathcal{L}_\lambda} \colon \lambda \in \Lambda\} < \infty\} \tag{3.11}$$

with coordinate-wise operations and the multiplication constant t_Λ is called the normed direct product.

Identify each $\mathcal{L}_\lambda$ with $\left\{(a_\mu)_{\mu\in\Lambda} \in \oplus_\Lambda \mathcal{L}_\mu \colon a_\mu = 0 \text{ for } \mu \neq \lambda\right\}$. The closed Lie ideal $\widehat{\mathcal{L}} = \widehat{\oplus}_\Lambda \mathcal{L}_\lambda$ of $\mathcal{L}$ generated by all Lie ideals $\mathcal{L}_\lambda$ is called the c_0-direct product.

If $\Lambda = \mathbb{N}$ then $\widehat{\mathcal{L}} = \widehat{\oplus}_{\mathbb{N}} \mathcal{L}_n = \{(a_n)_{n\in\mathbb{N}} \in \mathcal{L}\colon \|a_n\|_{\mathcal{L}_n} \to 0 \text{ as } n \to \infty\}$.

Proposition 3.13. *Let $\mathcal{L} = \oplus_\Lambda \mathcal{L}_\lambda$ and $\widehat{\mathcal{L}} = \widehat{\oplus}_\Lambda \mathcal{L}_\lambda$. If R is a balanced preradical then*

$$R\left(\widehat{\mathcal{L}}\right) = \widehat{\oplus}_\Lambda R(\mathcal{L}_\lambda) \subseteq R(\mathcal{L}) \subseteq \oplus_\Lambda R(\mathcal{L}_\lambda).$$

In particular, $\mathcal{L} \in \mathbf{Sem}(R)$ if and only if all $\mathcal{L}_\lambda \in \mathbf{Sem}(R)$.

Proof. Let $\mathcal{N}_\mu = \left\{(a_\lambda)_{\lambda\in\Lambda} \in \oplus_\Lambda \mathcal{L}_\lambda : a_\mu = 0\right\}$. Then $\mathcal{L} = \mathcal{N}_\mu \oplus \mathcal{L}_\mu$ for each $\mu \in \Lambda$. By Proposition 3.11(ii) 1), $R(\mathcal{L}) = R(\mathcal{N}_\mu) \oplus R(\mathcal{L}_\mu)$. Hence

$$R(\mathcal{L}) = \cap_{\mu\in\Lambda}(R(\mathcal{N}_\mu) \oplus R(\mathcal{L}_\mu)) \subseteq \cap_{\mu\in\Lambda}(\mathcal{N}_\mu \oplus R(\mathcal{L}_\mu)) = \oplus_\Lambda R(\mathcal{L}_\lambda).$$

As all $\mathcal{L}_\lambda \lhd \widehat{\mathcal{L}} \lhd \mathcal{L}$ and R is balanced, all $R(\mathcal{L}_\lambda) \subseteq R\left(\widehat{\mathcal{L}}\right) \subseteq R(\mathcal{L})$. Hence

$$\widehat{\oplus}_\Lambda R(\mathcal{L}_\lambda) \subseteq R\left(\widehat{\mathcal{L}}\right) \subseteq R(\mathcal{L}) \subseteq \oplus_\Lambda R(\mathcal{L}_\lambda). \tag{3.12}$$

As $R\left(\widehat{\mathcal{L}}\right) \subseteq \widehat{\mathcal{L}}$ and $R\left(\widehat{\mathcal{L}}\right) \subseteq \oplus_\Lambda R\left(\mathcal{L}_\lambda\right)$, we have $R\left(\widehat{\mathcal{L}}\right) \subseteq (\oplus_\Lambda R(\mathcal{L}_\lambda)) \cap \widehat{\mathcal{L}} = \widehat{\oplus}_\Lambda R\left(\mathcal{L}_\lambda\right)$. Hence, by (3.12), $R\left(\widehat{\mathcal{L}}\right) = \widehat{\oplus}_\Lambda R\left(\mathcal{L}_\lambda\right)$. Together with (3.12) this gives us the complete proof. □

4. Construction of radicals from preradicals

In this section we consider various ways to improve preradicals, that is, to construct from them new preradicals with additional better properties (in particular, radicals). First we consider some operations on families of closed subspaces.

Let G be a family of closed subspaces of a Banach space X. Denote by $\sum_{Y\in G} Y$ the linear subspace of X that consists of all finite sums of elements from all $Y \in G$. Set

$$\mathfrak{p}\left(G\right) = X, \text{ if } G = \varnothing, \text{ and } \mathfrak{p}\left(G\right) = \bigcap_{Y\in G} Y, \text{ if } G \neq \varnothing, \tag{4.1}$$

$$\mathfrak{s}\left(G\right) = \{0\}, \text{ if } G = \varnothing, \text{ and } \mathfrak{s}\left(G\right) = \overline{\sum_{Y\in G} Y}, \text{ if } G \neq \varnothing. \tag{4.2}$$

Let f be a continuous linear map from X into a Banach space Z. Then

$$f(G) := \{\overline{f(Y)} : Y \in G\}.$$

is a family of closed subspaces in Z. As $f(\sum_{Y\in G} Y) = \sum_{Y\in G} f(Y)$ and f is continuous,

$$f(\mathfrak{s}(G)) = f\left(\overline{\sum_{Y\in G} Y}\right) \subseteq \overline{\sum_{Y\in G} f(Y)} \subseteq \overline{\sum_{Y\in G} \overline{f(Y)}} = \mathfrak{s}(f(G)), \tag{4.3}$$

$$f(\mathfrak{p}(G)) = f(\bigcap_{Y\in G} Y) \subseteq \bigcap_{Y\in G} f(Y) \subseteq \bigcap_{Y\in G} \overline{f(Y)} = \mathfrak{p}(f(G)). \tag{4.4}$$

4.1. **R-superposition series.** We shall now develop a Lie algebraic version of the Dixon's constructions of radicals (see [D]) (in pure algebra they are known as Baer procedures).

Let R be a preradical. For $\mathcal{L} \in \mathfrak{L}$, set $R^0\left(\mathcal{L}\right) = \mathcal{L}$, $R^1\left(\mathcal{L}\right) = R\left(\mathcal{L}\right)$,

$$R^{\alpha+1}\left(\mathcal{L}\right) = R\left(R^\alpha\left(\mathcal{L}\right)\right), \text{ for an ordinal } \alpha \tag{4.5}$$
$$\text{and } R^\alpha\left(\mathcal{L}\right) = \underset{\alpha'<\alpha}{\cap} R^{\alpha'}\left(\mathcal{L}\right), \text{ for a limit ordinal } \alpha.$$

By Corollary 3.3, this is a decreasing transfinite chain of characteristic Lie ideals of $\mathcal{L}$. It stabilizes at some ordinal β: $R^{\beta+1}\left(\mathcal{L}\right) = R^\beta\left(\mathcal{L}\right)$, where β is bounded by an ordinal that depends on cardinality of $\mathcal{L}$. Denote the smallest such β by $r_R^\circ\left(\mathcal{L}\right)$ and set

$$R^\circ\left(\mathcal{L}\right) = R^{r_R^\circ(\mathcal{L})}(\mathcal{L}), \text{ so that } R(R^\circ(\mathcal{L})) = R^\circ(\mathcal{L}), \text{ for all } \mathcal{L} \in \mathfrak{L}. \tag{4.6}$$

Lemma 4.1. *Let R and T be preradicals. If at least one of them is balanced and $R \leq T$, then $R^\alpha \leq T^\alpha$ for every α, and $R^\circ \leq T^\circ$.*

Proof. Follows by induction. Indeed, let $\mathcal{L}$ be a Banach Lie algebra and $R^\alpha \leq T^\alpha$ for some α. Since $R^\alpha\left(\mathcal{L}\right) \lhd T^\alpha\left(\mathcal{L}\right)$, it follows that $R\left(R^\alpha\left(\mathcal{L}\right)\right) \subseteq R\left(T^\alpha\left(\mathcal{L}\right)\right) \subseteq T^{\alpha+1}\left(\mathcal{L}\right)$ if R is balanced, and $R^{\alpha+1}\left(\mathcal{L}\right) \subseteq T\left(R^\alpha\left(\mathcal{L}\right)\right) \subseteq T\left(T^\alpha\left(\mathcal{L}\right)\right)$ if T is balanced.

If $R^{\alpha'}\left(\mathcal{L}\right) \subseteq T^{\alpha'}\left(\mathcal{L}\right)$ for all $\alpha' < \alpha$, then $R^\alpha\left(\mathcal{L}\right) \subseteq T^\alpha\left(\mathcal{L}\right)$ follows from (4.5). Taking $\alpha \geq \max\left\{r_R^\circ(\mathcal{L}), r_T^\circ(\mathcal{L})\right\}$, we obtain that $R^\circ\left(\mathcal{L}\right) \subseteq T^\circ\left(\mathcal{L}\right)$ for every Banach Lie algebra $\mathcal{L}$. □

Theorem 4.2. *Let R be a balanced preradical. Then*

(i) *R^α is a balanced preradical for each ordinal α.*
(ii) *R° is an under radical,* $\mathbf{Rad}(R) = \mathbf{Rad}(R^\circ)$ *and* $\mathbf{Sem}(R) \subseteq \mathbf{Sem}(R^\circ)$. *Moreover, R° is the largest under radical smaller than or equal to R. If R is lower stable then $R^\circ = R$.*
(iii) *If $\mathcal{L} = \oplus_\Lambda \mathcal{L}_\lambda$ is the normed direct product of $\left\{\mathcal{L}_\lambda\right\}_\Lambda$, then $r_R^\circ\left(\mathcal{L}\right) \leq \max_\Lambda r_R^\circ\left(\mathcal{L}_\lambda\right)$.*

Proof. (i) Let R^{α} be a balanced preradical for some α. Let us show that $R^{\alpha+1}$ is a balanced preradical. We have $f(R^{\alpha}(\mathcal{L})) \subseteq R^{\alpha}(\mathcal{M})$ for each morphism $f\colon \mathcal{L} \longrightarrow \mathcal{M}$. As f is a homomorphism, $R^{\alpha}(\mathcal{L}) \lhd \mathcal{L}$ and $f(\mathcal{L})$ is dense in $\mathcal{M}$, we have $\overline{f(R^{\alpha}(\mathcal{L}))} \lhd \mathcal{M}$. Hence $\overline{f(R^{\alpha}(\mathcal{L}))} \lhd R^{\alpha}(\mathcal{M})$ and

$$f(R^{\alpha+1}(\mathcal{L})) = f(R(R^{\alpha}(\mathcal{L}))) \subseteq R\left(\overline{f(R^{\alpha}(\mathcal{L}))}\right) \subseteq R(R^{\alpha}(\mathcal{M})) = R^{\alpha+1}(\mathcal{M}).$$

Thus $R^{\alpha+1}$ is a preradical. As R^{α} is balanced, $R^{\alpha}(I) \subseteq R^{\alpha}(\mathcal{L})$ if $I \lhd \mathcal{L}$. By Corollary 3.3(ii), $R^{\alpha}(I)$ is a Lie ideal of $\mathcal{L}$. Hence $R^{\alpha}(I) \lhd R^{\alpha}(\mathcal{L})$. Since R is balanced, we have $R^{\alpha+1}(I) = R(R^{\alpha}(I)) \subseteq R(R^{\alpha}(\mathcal{L})) = R^{\alpha+1}(\mathcal{L})$. Thus $R^{\alpha+1}$ is balanced.

Let α be a limit ordinal and $R^{\alpha'}$, $\alpha' < \alpha$, be balanced preradicals. For $I \lhd \mathcal{L}$, $R^{\alpha}(I) = \bigcap_{\alpha'<\alpha} R^{\alpha'}(I) \subseteq \bigcap_{\alpha'<\alpha} R^{\alpha'}(\mathcal{L}) = R^{\alpha}(\mathcal{L})$ and, by (4.4), for each morphism $f\colon \mathcal{L} \longrightarrow \mathcal{M}$,

$$f(R^{\alpha}(\mathcal{L})) = f\left(\bigcap_{\alpha'<\alpha} R^{\alpha'}(\mathcal{L})\right) \subseteq \bigcap_{\alpha'<\alpha} f(R^{\alpha'}(\mathcal{L})) \subseteq \bigcap_{\alpha'<\alpha} R^{\alpha'}(\mathcal{M}) = R^{\alpha}(\mathcal{M}).$$

Thus R^{α} are balanced preradicals for all α.

(ii) From (i) and from the definition of R° we have that R° is a balanced preradical. As $\{R^{\alpha}(\mathcal{L})\}$ is decreasing, $R^{\circ}(\mathcal{L}) \subseteq R(\mathcal{L})$ for all $\mathcal{L} \in \mathfrak{L}$, so that $R^{\circ} \le R$. From this and from the definition of R° it follows that $R(\mathcal{L}) = \mathcal{L} \iff R^{\circ}(\mathcal{L}) = \mathcal{L}$. Thus $\mathbf{Rad}(R) = \mathbf{Rad}(R^{\circ})$. If $R(\mathcal{L}) = \{0\}$, it follows from the construction that $R^{\circ}(\mathcal{L}) = \{0\}$. Hence $\mathbf{Sem}(R) \subseteq \mathbf{Sem}(R^{\circ})$.

By (4.6), $R^{\circ}(\mathcal{L}) \in \mathbf{Rad}(R) = \mathbf{Rad}(R^{\circ})$. Thus R° is lower stable. Hence R° is an under radical.

If R is lower stable, then R is an under radical. As $\mathbf{Rad}(R) = \mathbf{Rad}(R^{\circ})$, it follows from Proposition 3.7(ii) that $R = R^{\circ}$.

Let T be an under radical and $T \le R$. If $\mathcal{L} \in \mathbf{Rad}(T)$ then $\mathcal{L} = T(\mathcal{L}) \subseteq R(\mathcal{L}) \subseteq \mathcal{L}$. Hence $\mathcal{L} \in \mathbf{Rad}(R) = \mathbf{Rad}(R^{\circ})$. Thus $\mathbf{Rad}(T) \subseteq \mathbf{Rad}(R^{\circ})$. By Proposition 3.7(i), $T \le R^{\circ}$. Part (ii) is proved. Part (iii) follows from Proposition 3.13. □

Proposition 4.3. *Let R be a balanced preradical and let $I \lhd \mathcal{L}$.*

(i) *If $\mathcal{L} \in \mathbf{Sem}(R^{\circ})$ then $I \in \mathbf{Sem}(R^{\circ})$ and $r_R^{\circ}(I) \le r_R^{\circ}(\mathcal{L})$.*

(ii) *If $I, \mathcal{L}/I \in \mathbf{Sem}(R^{\circ})$, then $\mathcal{L} \in \mathbf{Sem}(R^{\circ})$ and $r_R^{\circ}(\mathcal{L}) \le r_R^{\circ}(\mathcal{L}/I) + r_R^{\circ}(I)$.*

Proof. The first assertions in (i) and (ii) follow from (iii) and (v) of Lemma 3.6, respectively.

(i) As R is balanced then, by Theorem 4.2(i), R^{α} is balanced for each ordinal α. Let $\beta = r_R^{\circ}(\mathcal{L})$. Then $R^{\beta}(I) \subseteq R^{\beta}(\mathcal{L}) = R^{\circ}(\mathcal{L}) = \{0\}$. Hence $r_R^{\circ}(I) \le \beta$.

(ii) Let $q\colon \mathcal{L} \to \mathcal{L}/I$ be the quotient map, $\gamma = r_R^{\circ}(I)$ and $\beta = r_R^{\circ}(\mathcal{L}/I)$. As

$$q(R^{\beta}(\mathcal{L})) \overset{(3.1)}{\subseteq} R^{\beta}(q(\mathcal{L})) = R^{\beta}(\mathcal{L}/I) = R^{\circ}(\mathcal{L}/I) = \{0\},$$

we have $R^{\beta}(\mathcal{L}) \subseteq I$. Hence $R^{\gamma}(R^{\beta}(\mathcal{L})) \subseteq R^{\gamma}(I) = R^{\circ}(I) = \{0\}$. Thus $r_R^{\circ}(\mathcal{L}) \le \beta + \gamma$. □

Note that the order of ordinal summands in Proposition 4.3(ii) is essential, since, generally speaking, $R^{\beta+\gamma}(\mathcal{L}) = R^{\gamma}(R^{\beta}(\mathcal{L})) \ne R^{\beta}(R^{\gamma}(\mathcal{L})) = R^{\gamma+\beta}(\mathcal{L})$, so that $\beta + \gamma \ne \gamma + \beta$.

4.2. **R-convolution series.** For each preradical R, denote by q_R the quotient morphism on $\mathfrak{L}$: $q_R\colon \mathcal{L} \longrightarrow \mathcal{L}/R(\mathcal{L})$ for all $\mathcal{L} \in \mathfrak{L}$. Define a product $R*T$ of preradicals R, T on $\mathfrak{L}$ by the formula

$$(R*T)(\mathcal{L}) = q_T^{-1}(R(q_T(\mathcal{L}))) \text{ for each } \mathcal{L} \in \mathfrak{L}. \tag{4.7}$$

Proposition 4.4. *Let R, T be preradicals. Then*

(i) *$R*T$ is a preradical and $T \le R*T$.*

(ii) *If R is balanced then $R*T$ is balanced.*

(iii) *If R is lower stable then $R*T$ is lower stable.*

(iv) *If S is another preradical and $S \le T$, then $R*S \le R*T$ and $S*R \le T*R$.*

Proof. (i) By the definition, for each $\mathcal{L} \in \mathfrak{L}$, we have that $R(q_T(\mathcal{L}))$ is a closed Lie ideal of $q_T(\mathcal{L})$. As q_T is a bounded epimorphism, $q_T^{-1}(R(q_T(\mathcal{L})))$ is a closed Lie ideal of $\mathcal{L}$.

Let $f\colon \mathcal{L} \longrightarrow \mathcal{M}$ be a morphism in $\overline{\mathbf{L}}$. Set $q = q_T|\mathcal{L}$ and $q_1 = q_T|\mathcal{M}$. For each $x \in \mathcal{L}$, set $h(x) = q_1(f(x))$. Then h is a bounded homomorphism from $\mathcal{L}$ into $\mathcal{M}/T(\mathcal{M})$ with dense image. As T is a preradical, $f(T(\mathcal{L})) \subseteq T(\mathcal{M})$. Therefore, for each $a \in T(\mathcal{L})$,

$$h(x+a) = q_1(f(x) + f(a)) = q_1(f(x)).$$

Thus h generates a bounded homomorphism $\widetilde{h}\colon q(\mathcal{L}) = \mathcal{L}/T(\mathcal{L}) \longrightarrow \mathcal{M}/T(\mathcal{M}) = q_1(\mathcal{M})$ with dense image and $\widetilde{h}q = q_1 f$. Then $(R * T)(\mathcal{L}) = q_T^{-1}(R(q_T(\mathcal{L}))) = q^{-1}(R(q(\mathcal{L})))$, so that

$$q_1 f((R*T)(\mathcal{L})) = \widetilde{h}q(q^{-1}(R(q(\mathcal{L})))) = \widetilde{h}(R(q(\mathcal{L}))) \subseteq R(q_1(\mathcal{M})).$$

Therefore $f((R*T)(\mathcal{L})) = q_1^{-1}(R(q_1(\mathcal{M}))) = (R*T)(\mathcal{M})$. Thus $R*T$ is a preradical.

Clearly, $T(\mathcal{L}) \subseteq q_T^{-1}(R(q_T(\mathcal{L}))) = (R*T)(\mathcal{L})$, for each $\mathcal{L} \in \mathfrak{L}$, so that $T \le R*T$.

(ii) For $I \lhd \mathcal{L} \in \mathfrak{L}$, we have $q_T(I) \lhd q_T(\mathcal{L})$. If R is balanced, $R(q_T(I)) \subseteq R(q_T(\mathcal{L}))$. Hence

$$(R*T)(I) = q_T^{-1}(R(q_T(I))) \subseteq q_T^{-1}(R(q_T(\mathcal{L}))) = (R*T)(\mathcal{L}).$$

Thus the preradical $R*T$ is balanced.

(iii) If R is low stable, $R(R(\mathcal{L})) = R(\mathcal{L})$ for all $\mathcal{L} \in \mathfrak{L}$. Then $R*T$ is low stable, as

$$\begin{aligned}(R*T)((R*T)(\mathcal{L})) &= q_T^{-1}(R(q_T(q_T^{-1}(R(q_T(\mathcal{L})))))) = q_T^{-1}(R(R(q_T(\mathcal{L})))) \\ &= q_T^{-1}(R(q_T(\mathcal{L}))) = (R*T)(\mathcal{L}).\end{aligned}$$

(iv) Let $S \le T$ and $\mathcal{L} \in \mathfrak{L}$. Then $S(\mathcal{L}) \subseteq T(\mathcal{L})$. Hence there exists a quotient homomorphism $p\colon q_S(\mathcal{L}) = \mathcal{L}/S(\mathcal{L}) \longrightarrow \mathcal{L}/T(\mathcal{L}) = q_T(\mathcal{L})$, such that $q_T = pq_S$. Therefore

$$q_T((R*S)(\mathcal{L})) = pq_S(q_S^{-1}(R(q_S(\mathcal{L})))) = p(R(q_S(\mathcal{L}))) \subseteq R(pq_S(\mathcal{L})) = R(q_T(\mathcal{L})).$$

Thus $(R*S)(\mathcal{L}) \subseteq q_T^{-1}(R(q_T(\mathcal{L}))) = (R*T)(\mathcal{L})$. Hence $R*S \le R*T$.

As $S \le T$, we have $S(q_R(\mathcal{L})) \subseteq T(q_R(\mathcal{L}))$. Therefore

$$(S*R)(\mathcal{L}) = q_R^{-1}(S(q_R(\mathcal{L}))) \subseteq q_R^{-1}(T(q_R(\mathcal{L}))) = (T*R)(\mathcal{L}).$$

Thus $S*R \le T*R$. $\square$

For each preradical R, we will define now an upper stable preradical R^* in the following way. For $\mathcal{L} \in \mathfrak{L}$, set $R^{(0)}(\mathcal{L}) = \{0\}$, $R^{(1)}(\mathcal{L}) = R(\mathcal{L})$,

$$R^{(\alpha+1)}(\mathcal{L}) = (R * R^{(\alpha)})(\mathcal{L}), \text{ for an ordinal } \alpha, \tag{4.8}$$

By Proposition 4.4(i), we have $R^{(\alpha)}(\mathcal{L}) \subseteq R^{(\alpha+1)}(\mathcal{L})$. Hence we can define

$$R^{(\alpha)}(\mathcal{L}) = \overline{\underset{\alpha'<\alpha}{\cup} R^{(\alpha')}(\mathcal{L})}, \text{ for a limit ordinal } \alpha. \tag{4.9}$$

Then $\{R^{(\alpha)}(\mathcal{L})\}$ is an increasing transfinite chain. By Corollary 3.3, it consists of characteristic Lie ideals of $\mathcal{L}$. Since all α are bounded by an ordinal that depends on cardinality of $\mathcal{L}$, the chain stabilizes at some ordinal β: $R^{(\beta+1)}(\mathcal{L}) = R^{(\beta)}(\mathcal{L})$. Set

$$R^*(\mathcal{L}) = R^{\beta}(\mathcal{L}), \text{ so that } R*R^* = R^*. \tag{4.10}$$

Theorem 4.5. (i) *Let R be a preradical. Then R^* is an upper stable preradical,*

$$R \le R^*, \quad \mathbf{Sem}(R) = \mathbf{Sem}(R^*) \textit{ and } \mathbf{Rad}(R) \subseteq \mathbf{Rad}(R^*).$$

If R is balanced, then R^* is an over radical. Moreover, R^* is the smallest over radical larger than or equal to R. If R is upper stable then $R^* = R$.

(ii) *Let R and T be preradicals. If $R \le T$, then $R^{(\alpha)} \le T^{(\alpha)}$ for each α, and $R^* \le T^*$.*

Proof. Let $R^{(\alpha)}$ be a preradical for some α. By Proposition 4.4(i), $R^{(\alpha+1)} = R * R^{(\alpha)}$ is a preradical for an ordinal α. Let α be a limit ordinal and let $R^{(\alpha')}$, for $\alpha' < \alpha$, be preradicals. For each morphism $f\colon \mathcal{L} \longrightarrow \mathcal{M}$ it follows from (4.3) that

$$f(R^{(\alpha)}(\mathcal{L})) = f\left(\overline{\underset{\alpha'<\alpha}{\cup} R^{(\alpha')}(\mathcal{L})}\right) \subseteq \overline{\underset{\alpha'<\alpha}{\cup} f(R^{(\alpha')}(\mathcal{L}))} \subseteq \overline{\underset{\alpha'<\alpha}{\cup} R^{(\alpha')}(\mathcal{M})} = R^{\alpha}(\mathcal{M}).$$

Thus $R^{(\alpha)}$ are preradicals for all α, so that R^* is a preradical.

By (4.8), (4.9) and Proposition 4.4(i), we have $R \le R^*$. Hence $R^*(\mathcal{L}) = \{0\}$ implies $R(\mathcal{L}) = \{0\}$. If $R(\mathcal{L}) = \{0\}$, it follows from (4.7) – (4.9) that $R^*(\mathcal{L}) = \{0\}$. Thus $\mathbf{Sem}(R) = \mathbf{Sem}(R^*)$.

If $R(\mathcal{L}) = \mathcal{L}$, it follows that all $R^{(\alpha)}(\mathcal{L}) = \mathcal{L}$, so that $R^*(\mathcal{L}) = \mathcal{L}$. Hence $\mathbf{Rad}(R) \subseteq \mathbf{Rad}(R^*)$.

Set $q = q_{R^*}$. As $R^* = R * R^*$ (see (4.10)), we have from (4.7) that $R^*(\mathcal{L}) = (R * R^*)(\mathcal{L}) = q^{-1}(R(q(\mathcal{L})))$. Hence $q(R^*(\mathcal{L})) = R(q(\mathcal{L}))$. As $q\colon \mathcal{L} \longrightarrow \mathcal{L}/R^*(\mathcal{L})$, we have $q(R^*(\mathcal{L})) = \{0\}$. Hence $R(q(\mathcal{L})) = 0$. Thus $q(\mathcal{L})$ is R-semisimple, so that $q(\mathcal{L})$ is R^*-semisimple by the above argument. Hence $R^*(\mathcal{L}/R^*(\mathcal{L})) = 0$, whence R^* is upper stable.

By (4.10), $R^* = R * R^*$. Hence, if R is balanced, it follows from Proposition 4.4(ii) that R^* is balanced. Thus R^* is an over radical.

Let T be an over radical such that $R \le T$. If $\mathcal{L} \in \mathbf{Sem}(T)$ then $R(\mathcal{L}) \subseteq T(\mathcal{L}) = \{0\}$. Hence $\mathcal{L} \in \mathbf{Sem}(R) = \mathbf{Sem}(R^*)$. Thus $\mathbf{Sem}(T) \subseteq \mathbf{Sem}(R^*)$. By Proposition 3.7(iii), $R^* \le T$.

Let R be upper stable. As R^* is upper stable and $\mathbf{Sem}(R) = \mathbf{Sem}(R^*)$, it follows from Proposition 3.7(iv) that $R = R^*$. Part (i) is proved.

Part (ii) follows by induction. Indeed, let $R^{(\alpha)} \le T^{(\alpha)}$ for some α. As $R \le T$, we have from Proposition 4.4(iv) $R^{(\alpha+1)} = R * R^{(\alpha)} \le R * T^{(\alpha)} \le T * T^{(\alpha)} = T^{(\alpha+1)}$. Let $\mathcal{L}$ be a Banach Lie algebra. If $R^{(\alpha')}(\mathcal{L}) \subseteq T^{(\alpha')}(\mathcal{L})$ for all $\alpha' < \alpha$, then $R^{(\alpha)}(\mathcal{L}) \subseteq T^{(\alpha)}(\mathcal{L})$ follows from (4.9). Taking $\alpha \ge \max\{r_R^*(\mathcal{L}), r_T^*(\mathcal{L})\}$, we have $R^*(\mathcal{L}) \subseteq T^*(\mathcal{L})$ for each Banach Lie algebra $\mathcal{L}$. $\square$

The following theorem gives sufficient conditions for R° and R^* to be radicals. It is similar to the result proved in [D] for the category of associative normed algebras.

Theorem 4.6. (i) *If R is an under radical then R^* is the smallest radical larger than or equal to R.*

(ii) *If R is an over radical then R° is the largest radical smaller than or equal to R.*

Proof. (i) By (4.10), $R^* = R * R^*$. As R is lower stable, Proposition 4.4(iii) implies that R^* is lower stable. Hence, by Theorem 4.5(i), R^* is a radical, $R \le R^*$ and R^* is the smallest over radical larger than or equal to R. Hence R^* is the smallest radical larger than or equal to R.

(ii) Let R^α be upper stable for some α: $R^\alpha(\mathcal{L}/R^\alpha(\mathcal{L})) = \{0\}$ for all $\mathcal{L} \in \mathfrak{L}$. As $R^{\alpha+1}(\mathcal{L}) \subseteq R^\alpha(\mathcal{L})$, there is the quotient map $q : \mathcal{L}/R^{\alpha+1}(\mathcal{L}) \longrightarrow \mathcal{L}/R^\alpha(\mathcal{L})$. Since R^α is a preradical, $q(R^\alpha(\mathcal{L}/R^{\alpha+1}(\mathcal{L}))) \subseteq R^\alpha(\mathcal{L}/R^\alpha(\mathcal{L})) = \{0\}$. Hence $R^\alpha(\mathcal{L}/R^{\alpha+1}(\mathcal{L})) \subseteq \ker(q) = R^\alpha(\mathcal{L})/R^{\alpha+1}(\mathcal{L})$. As R is upper stable,

$$R(R^\alpha(\mathcal{L})/R^{\alpha+1}(\mathcal{L})) = R(R^\alpha(\mathcal{L})/R(R^\alpha(\mathcal{L}))) = \{0\}.$$

Therefore, as R is balanced,

$$R^{\alpha+1}\left(\mathcal{L}/R^{\alpha+1}(\mathcal{L})\right) = R\left(R^\alpha\left(\mathcal{L}/R^{\alpha+1}(\mathcal{L})\right)\right) \subseteq R\left(R^\alpha(\mathcal{L})/R^{\alpha+1}(\mathcal{L})\right) = \{0\}.$$

Thus $R^{\alpha+1}(\mathcal{L})$ is upper stable.

Let α be a limit ordinal. For all $\alpha' < \alpha$, $R^{\alpha'}\left(\mathcal{L}/R^{\alpha'}(\mathcal{L})\right) = \{0\}$ and $R^\alpha(\mathcal{L}) \subseteq R^{\alpha'}(\mathcal{L})$ for each $\mathcal{L} \in \mathfrak{L}$. Let q be the quotient map $q : \mathcal{L}/R^\alpha(\mathcal{L}) \longrightarrow \mathcal{L}/R^{\alpha'}(\mathcal{L})$. Since $R^{\alpha'}$ is a preradical, $q(R^{\alpha'}(\mathcal{L}/R^\alpha(\mathcal{L}))) \subseteq R^{\alpha'}(q(\mathcal{L}/R^\alpha(\mathcal{L}))) = R^{\alpha'}\left(\mathcal{L}/R^{\alpha'}(\mathcal{L})\right) = \{0\}$. Hence $R^{\alpha'}(\mathcal{L}/R^\alpha(\mathcal{L})) \subseteq \ker(q) = R^{\alpha'}(\mathcal{L})/R^\alpha(\mathcal{L})$. Therefore, as $R^\alpha(\mathcal{L}) = \underset{\alpha'<\alpha}{\cap} R^{\alpha'}(\mathcal{L})$, we have

$$R^\alpha(\mathcal{L}/R^\alpha(\mathcal{L})) = \underset{\alpha'<\alpha}{\cap} R^{\alpha'}(\mathcal{L}/R^\alpha(\mathcal{L})) \subseteq \underset{\alpha'<\alpha}{\cap}\left\{R^{\alpha'}(\mathcal{L})/R^\alpha(\mathcal{L})\right\} = \{0\}.$$

Thus R^α is upper stable for all α, so that R° is upper stable. Hence, by Theorem 4.2, R° is a radical, $R^\circ \le R$ and R° is the largest under radical smaller than or equal to R. $\square$

4.3. **Construction of under radicals by subideals.** Let $\mathcal{L} \in \mathfrak{L}$. Recall that a closed Lie subalgebra I of $\mathcal{L}$ is a Lie subideal ($I \lll \mathcal{L}$), if there is a chain of closed Lie subalgebras J_0,..., J_n of $\mathcal{L}$ such that $I = J_0 \lhd J_1 \lhd \cdots \lhd J_n = \mathcal{L}$. Let R be a preradical. Set

$$\mathrm{Sub}\,(\mathcal{L}, R) = \{I \lll \mathcal{L} : R\,(I) = I\} \text{ and } R^{\mathrm{s}}\,(\mathcal{L}) = \mathfrak{s}\,(\mathrm{Sub}\,(\mathcal{L}, R)) \text{ (see (4.2)).} \tag{4.11}$$

The subideals in $\mathrm{Sub}(\mathcal{L}, R)$ are called R-radical. Clearly, $R^{\mathrm{s}}\,(\mathcal{L})$ is a closed subspace of $\mathcal{L}$.

Lemma 4.7. *Let R and T be preradicals. If $R \leq T$ then $R^{\mathrm{s}} \leq T^{\mathrm{s}}$.*

Proof. Let $\mathcal{L}$ be a Banach Lie algebra and $I \in \mathrm{Sub}\,(\mathcal{L}, R)$. As $\mathbf{Rad}\,(R) \subseteq \mathbf{Rad}\,(T)$, we have $I \in \mathrm{Sub}\,(\mathcal{L}, T)$. Hence $R^{\mathrm{s}}\,(\mathcal{L}) \subseteq T^{\mathrm{s}}\,(\mathcal{L})$. Thus $R^{\mathrm{s}} \leq T^{\mathrm{s}}$. □

Theorem 4.8. *Let R be a preradical in $\overline{\mathbf{L}}$. Then*

(i) *R^{s} is a balanced, lower stable preradical, so that R^{s} is an under radical in $\overline{\mathbf{L}}$.*
(ii) *If R is balanced, then $R^{\mathrm{s}} \leq R$ (see (3.8)) and R^{s} is the largest under radical smaller than or equal to R.*
(iii) *If R is lower stable, then $R \leq R^{\mathrm{s}}$ and R^{s} is the smallest under radical larger than or equal to R.*
(iv) *If R is an under radical then $R = R^{\mathrm{s}}$.*

Proof. (i) Let $f\colon\, \mathcal{L} \longrightarrow \mathcal{M}$ be a morphism in $\overline{\mathbf{L}}$, $I \in \mathrm{Sub}\,(\mathcal{L}, R)$ and $I = J_0 \lhd \cdots \lhd J_n = \mathcal{L}$ for some closed Lie algebras J_i in $\mathcal{L}$. Then $\overline{f(I)} = \overline{f(J_0)} \lhd \cdots \lhd \overline{f(J_n)} = \mathcal{M}$ and all $\overline{f(J_i)}$ are closed Lie algebras in $\mathcal{M}$. Hence $\overline{f(I)} \lll \mathcal{M}$. As $I = R(I)$, we have from (3.1) that $f(I) = f(R(I)) \subseteq R\left(\overline{f(I)}\right) \subseteq \overline{f(I)}$. As $R\left(\overline{f(I)}\right)$ is closed, $R\left(\overline{f(I)}\right) = \overline{f(I)}$. Thus $\overline{f(I)} \in \mathrm{Sub}\,(\mathcal{M}, R)$ and

$$f\left(\sum_{I \in \mathrm{Sub}(\mathcal{L},R)} I\right) = \sum_{I \in \mathrm{Sub}(\mathcal{L},R)} f(I) \subseteq \sum_{J \in \mathrm{Sub}(\mathcal{M},R)} J,$$

so that $f(R^{\mathrm{s}}\,(\mathcal{L})) \subseteq R^{\mathrm{s}}\,(\mathcal{M})$. Therefore R^{s} is a preradical (see (3.1)).

Let $K \lhd \mathcal{L}$. If $I \in \mathrm{Sub}\,(K, R)$ then $I = J_0 \lhd \cdots \lhd J_n = K$ for some closed Lie algebras J_i, whence $I \in \mathrm{Sub}\,(\mathcal{L}, R)$. Thus $\mathrm{Sub}\,(K, R) \subseteq \mathrm{Sub}\,(\mathcal{L}, R)$. Hence R^{s} is balanced (see (3.5)), since by (4.11),

$$R^{\mathrm{s}}\,(K) = \overline{\sum_{I \in \mathrm{Sub}(K,R)} I} \subseteq \overline{\sum_{I \in \mathrm{Sub}(\mathcal{L},R)} I} = R^{\mathrm{s}}\,(\mathcal{L})\,.$$

Set $K = R^{\mathrm{s}}\,(\mathcal{L})$. If $I \in \mathrm{Sub}\,(\mathcal{L}, R)$ then $I = J_0 \lhd \cdots \lhd J_n = \mathcal{L}$. By (4.11), $I \subseteq K$. Hence $I = (J_0 \cap K) \lhd \cdots \lhd (J_n \cap K) = K$, so that $I \in \mathrm{Sub}\,(K, R)\,$. Thus $\mathrm{Sub}\,(\mathcal{L}, R) = \mathrm{Sub}\,(K, R)$. Hence, by (4.11), $R^{\mathrm{s}}\,(R^{\mathrm{s}}\,(\mathcal{L})) = R^{\mathrm{s}}\,(K) = R^{\mathrm{s}}\,(\mathcal{L})\,$. Thus (see (3.3)) R^{s} is lower stable.

(iv) Let R be balanced. If $I \in \mathrm{Sub}\,(\mathcal{L}, R)$ then $I = J_0 \lhd \cdots \lhd J_n = \mathcal{L}$ and $I = R(I) = R(J_0) \subseteq \ldots \subseteq R(J_n) = R\,(\mathcal{L})$. Hence, as $R\,(\mathcal{L})$ is closed, it follows from (4.11) that $R^{\mathrm{s}}\,(\mathcal{L}) \subseteq R\,(\mathcal{L})$ .Thus $R^{\mathrm{s}} \leq R$. This also proves the first statement of (ii).

Let R be lower stable. Then $R(R(\mathcal{L})) = R(\mathcal{L})$ for all $\mathcal{L} \in \mathfrak{L}$, so that $R(\mathcal{L}) \in \mathrm{Sub}\,(\mathcal{L}, R)\,$. Hence, by (4.11), $R(\mathcal{L}) \subseteq R^{\mathrm{s}}(\mathcal{L})$. Thus $R \leq R^{\mathrm{s}}$. This also proves the first statement of (iii).

So if R is balanced and lower stable then $R = R^{\mathrm{s}}$ that proves (iv).

Let us finish the proof of (ii) and (iii).

(ii) Let R be balanced, let Q be an under radical and $Q \leq R$. If I is a Q-radical subideal of $\mathcal{L}$, then I is an R-radical subideal of $\mathcal{L}$. Hence $Q^{\mathrm{s}} \leq R^{\mathrm{s}}$. It follows from (iv) that $Q = Q^{\mathrm{s}}$. Therefore R^{s} is the largest under radical smaller than or equal to R.

(iii) Let R lower stable, let T be an under radical and $R \leq T$. If $I \in \mathrm{Sub}\,(\mathcal{L}, R)$ then $I = R(I) \subseteq T(I) \subseteq I$. Hence $I = T(I)$, so that $I \in \mathrm{Sub}\,(\mathcal{L}, T)$. Thus $\mathrm{Sub}\,(\mathcal{L}, R) \subseteq \mathrm{Sub}\,(\mathcal{L}, T)$. Using (4.11), we have $R^{\mathrm{s}}\,(\mathcal{L}) \subseteq T^{\mathrm{s}}\,(\mathcal{L})$ for each $\mathcal{L} \in \mathfrak{L}$. By (iv), $T = T^{\mathrm{s}}$, whence $R^{\mathrm{s}} \leq T$. Therefore R^{s} is the smallest under radical larger than or equal to R. □

Theorem 4.8(i) and (iv) yield that $R^{\mathrm{ss}} = R^{\mathrm{s}}$ for each preradical R.

Corollary 4.9. *Let R be a preradical in $\overline{\mathbf{L}}$.*

(i) *If R is lower stable then $R \le (R^{\mathbf{s}})^*$ and $(R^{\mathbf{s}})^*$ is the smallest radical larger than or equal to R.*
(ii) *If R is balanced then $R^{\mathbf{s}} = R^\circ \le R \le R^*$, $(R^\circ)^*$ and $(R^*)^\circ$ are radicals, and $(R^\circ)^* \le (R^*)^\circ$.*

Proof. (i) If R is lower stable then $R \le R^{\mathbf{s}}$ and $R^{\mathbf{s}}$ is an under radical by Theorem 4.8(iii), and $R^{\mathbf{s}*}$ is a radical by Theorem 4.6(i). By definition, $R^{\mathbf{s}} \le R^{\mathbf{s}*}$. So $R \le R^{\mathbf{s}*}$.

Let T be a radical and $R \le T$. Then $R^{\mathbf{s}} \le T^{\mathbf{s}}$ by Lemma 4.7. As T is an under radical, $T^{\mathbf{s}} = T$ by Theorem 4.8(iv). Therefore $R^{\mathbf{s}} \le T$. By Theorem 4.5(ii), $(R^{\mathbf{s}})^* \le T^*$. As T is upper stable, $T^* = T$ by Theorem 4.5(i). Thus $(R^{\mathbf{s}})^*$ is the smallest radical larger than or equal to R.

(ii) Let R be balanced. Then $R^{\mathbf{s}} = R^\circ$ by Theorems 4.2(ii) and 4.8(ii).

It follows from Theorems 4.2, 4.5 and 4.6 that $(R^\circ)^*$ and $(R^*)^\circ$ are radicals. Further, by Theorems 4.2 and 4.5, $R^\circ \le R \le R^*$. By Lemma 4.1, $(R^\circ)^\circ \le (R^*)^\circ$. As R° is lower stable, $R^\circ = (R^\circ)^\circ$ by Theorem 4.2. Therefore $R^\circ \le (R^*)^\circ$. Since, by Theorem 4.6(i), $(R^\circ)^*$ is a smallest radical larger than or equal to R°, we have $(R^\circ)^* \le (R^*)^\circ$. $\square$

5. Construction of preradicals from multifunctions

Some important preradicals in $\overline{\mathbf{L}}$ and its subcategories arise from subspace-multifunctions on $\mathfrak{L}$; this link we will study now.

Let F, G be non-empty families of subspaces of X. We write

$$F \overrightarrow{\subset} G \text{ if, for each } Y \in F, \text{ there is } Z \in G \text{ such that } Y \subseteq Z; \tag{5.1}$$
$$G \overleftarrow{\subset} F \text{ if, for each } Y \in F, \text{ there is } Z \in G \text{ such that } Z \subseteq Y.$$

We assume that $\varnothing \overrightarrow{\subset} G$ and $G \overleftarrow{\subset} \varnothing$. By (5.1), if $F \subseteq G$ then $F \overrightarrow{\subset} G$ and $G \overleftarrow{\subset} F$. It follows from (4.1), (4.2) and (5.1) that

$$\text{if } F \overrightarrow{\subset} G \text{ then } \mathfrak{s}(F) \subseteq \mathfrak{s}(G); \qquad \text{if } G \overleftarrow{\subset} F \text{ then } \mathfrak{p}(G) \subseteq \mathfrak{p}(F). \tag{5.2}$$

If $G \neq \varnothing$ then $\{\{0\}\} \overleftarrow{\subset} G \overrightarrow{\subset} \{X\}$. For one-element families $F = \{Y\}$ and $G = \{Z\}$, both relations coincide with inclusion: if $\{Y\} \overrightarrow{\subset} \{Z\}$, or $\{Y\} \overleftarrow{\subset} \{Z\}$, then $Y \subseteq Z$.

Definition 5.1. *If, for each $\mathcal{L} \in \mathfrak{L}$, a family $\Gamma_{\mathcal{L}}$ of closed subspaces (Lie algebras, Lie ideals) of $\mathcal{L}$ is given, we say that $\Gamma = \{\Gamma_{\mathcal{L}}\}$ is a subspace (Lie algebra, Lie ideal)-multifunction on $\mathfrak{L}$.*

Let $\Gamma = \{\Gamma_{\mathcal{L}}\}$ be a subspace-multifunction. Making use of (4.1), set

$$P_\Gamma(\mathcal{L}) = \mathfrak{p}(\Gamma_{\mathcal{L}}) \text{ and } S_\Gamma(\mathcal{L}) = \mathfrak{s}(\Gamma_{\mathcal{L}}), \text{ for each } \mathcal{L} \in \mathfrak{L}. \tag{5.3}$$

If, for example, $\Gamma_{\mathcal{L}}$ is a singleton $\{\phi(\mathcal{L})\}$ for each $\mathcal{L} \in \mathfrak{L}$, then $S_\Gamma(\mathcal{L}) = P_\Gamma(\mathcal{L}) = \phi(\mathcal{L})$.

Definition 5.2. *Let Γ be a subspace-multifunction on $\mathfrak{L}$. If, for each morphism $f: \mathcal{L} \longrightarrow \mathcal{M}$, the family $f(\Gamma_{\mathcal{L}}) = \{\overline{f(Y)} : Y \in \Gamma_{\mathcal{L}}\}$ of closed subspaces of $\mathcal{M}$ satisfies*
(i) *$f(\Gamma_{\mathcal{L}}) \overrightarrow{\subset} \Gamma_{\mathcal{M}}$ then the multifunction Γ is called* direct;
(ii) *$f(\Gamma_{\mathcal{L}}) \subseteq \Gamma_{\mathcal{M}}$ then the multifunction Γ is called* strictly direct;
(iii) *$f(\Gamma_{\mathcal{L}}) \overleftarrow{\subset} \Gamma_{\mathcal{M}}$ then the multifunction Γ is called* inverse.

Proposition 5.3. (i) *If Γ is a direct multifunction then S_Γ is a preradical in $\overline{\mathbf{L}}$.*
(ii) *If Γ is an inverse multifunction then P_Γ is a preradical in $\overline{\mathbf{L}}$.*

Proof. If Γ is direct then $f(\Gamma_{\mathcal{L}}) \overrightarrow{\subset} \Gamma_{\mathcal{M}}$ for each morphism $f\colon \mathcal{L} \longrightarrow \mathcal{M}$. Therefore

$$f(S_\Gamma(\mathcal{L})) \overset{(5.3)}{=} f(\mathfrak{s}(\Gamma_{\mathcal{L}})) \overset{(4.3)}{\subseteq} \mathfrak{s}(f(\Gamma_{\mathcal{L}})) \overset{(5.2)}{\subseteq} \mathfrak{s}(\Gamma_{\mathcal{M}}) \overset{(5.3)}{=} S_\Gamma(\mathcal{M}).$$

If Γ is inverse then $f(\Gamma_{\mathcal{L}}) \overleftarrow{\subset} \Gamma_{\mathcal{M}}$ for each morphism $f\colon \mathcal{L} \longrightarrow \mathcal{M}$. Therefore

$$f(P_\Gamma(\mathcal{L})) \overset{(5.3)}{=} f(\mathfrak{p}(\Gamma_{\mathcal{L}})) \overset{(4.4)}{\subseteq} \mathfrak{p}(f(\Gamma_{\mathcal{L}})) \overset{(5.2)}{\subseteq} \mathfrak{p}(\Gamma_{\mathcal{M}}) \overset{(5.3)}{=} P_\Gamma(\mathcal{M}).$$

Take $\mathcal{M} = \mathcal{L}$. Considering inner automorphisms $f = \exp t(\mathrm{ad}(a))$ for $a \in \mathcal{L}$, $t \in \mathbb{C}$, we get that $P_\Gamma(\mathcal{L})$ and $S_\Gamma(\mathcal{L})$ are ideals of $\mathcal{L}$. Thus we have from (3.1) that S_Γ and P_Γ are preradicals. $\square$

If Γ is a direct subspace-multifunction on $\mathfrak{L}$, set $I_{\mathcal{L}} = S_\Gamma(\mathcal{L})$. If Γ is an inverse subspace-multifunction on $\mathfrak{L}$, set $I_{\mathcal{L}} = P_\Gamma(\mathcal{L})$. For $J \lhd \mathcal{L}$, set

$$\Gamma_{\mathcal{L}} \cap J = \{L \cap J : L \in \Gamma_{\mathcal{L}}\}. \tag{5.4}$$

Definition 5.4. *A direct (respectively, inverse) subspace-multifunction Γ is called*

(i) balanced *if $\Gamma_J \overrightarrow{\subset} \Gamma_{\mathcal{L}}$ (respectively, $\Gamma_J \overleftarrow{\subset} \Gamma_{\mathcal{L}}$) for all $J \lhd \mathcal{L} \in \mathfrak{L}$;*
(ii) lower stable *if $\Gamma_{\mathcal{L}} \overrightarrow{\subset} \Gamma_{I_{\mathcal{L}}}$ (respectively, $\Gamma_{\mathcal{L}} \overleftarrow{\subset} \Gamma_{I_{\mathcal{L}}}$) for all $\mathcal{L} \in \mathfrak{L}$;*
(iii) upper stable *if $\Gamma_{\mathcal{L}/I_{\mathcal{L}}} = \{\{0\}\}$ (respectively, $\mathfrak{p}(\Gamma_{\mathcal{L}/I_{\mathcal{L}}}) = \{0\}$) for all $\mathcal{L} \in \mathfrak{L}$.*

Theorem 5.5. *Let Γ be a direct* (*respectively, inverse*) *subspace-multifunction on $\mathfrak{L}$.*

(i) *If Γ is lower stable then S_Γ (respectively, P_Γ) is a lower stable preradical.*
(ii) *If Γ is balanced then S_Γ (respectively, P_Γ) is a balanced preradical.*
(iii) *If Γ is upper stable then S_Γ (respectively, P_Γ) is an upper stable preradical.*

Proof. By Proposition 5.3, S_Γ is a preradical if Γ is direct, and P_Γ is a preradical if Γ is inverse.

(i) Let Γ be lower stable. If Γ is direct, $\Gamma_{\mathcal{L}} \overrightarrow{\subset} \Gamma_{I_{\mathcal{L}}}$ for all $\mathcal{L} \in \mathfrak{L}$, where $I_{\mathcal{L}} = S_\Gamma(\mathcal{L})$. Hence

$$S_\Gamma(\mathcal{L}) \overset{(5.3)}{=} \mathfrak{s}(\Gamma_{\mathcal{L}}) \overset{(5.2)}{\subseteq} \mathfrak{s}(\Gamma_{I_{\mathcal{L}}}) \overset{(5.3)}{=} S_\Gamma(I_{\mathcal{L}}) = S_\Gamma(S_\Gamma(\mathcal{L})).$$

If Γ is inverse, $\Gamma_{\mathcal{L}} \overleftarrow{\subset} \Gamma_{I_{\mathcal{L}}}$ for all $\mathcal{L} \in \mathfrak{L}$, where $I_{\mathcal{L}} = P_\Gamma(\mathcal{L})$. Hence

$$P_\Gamma(\mathcal{L}) \overset{(5.3)}{=} \mathfrak{p}(\Gamma_{\mathcal{L}}) \overset{(5.2)}{\subseteq} \mathfrak{p}(\Gamma_{I_{\mathcal{L}}}) \overset{(5.3)}{=} P_\Gamma(I_{\mathcal{L}}) = P_\Gamma(P_\Gamma(\mathcal{L})).$$

Thus (see (3.3)) S_Γ and P_Γ are lower stable.

(ii) Let Γ be balanced. If Γ is direct, $\Gamma_J \overrightarrow{\subset} \Gamma_{\mathcal{L}}$ for all $J \lhd \mathcal{L} \in \mathfrak{L}$. Hence $S_\Gamma(J) \overset{(5.3)}{=} \mathfrak{s}(\Gamma_J) \overset{(5.2)}{\subseteq} \mathfrak{s}(\Gamma_{\mathcal{L}}) \overset{(5.3)}{=} S_\Gamma(\mathcal{L})$.

If Γ is inverse, $\Gamma_J \overleftarrow{\subset} \Gamma_{\mathcal{L}}$ for all $J \lhd \mathcal{L} \in \mathfrak{L}$. Hence $P_\Gamma(J) \overset{(5.3)}{=} \mathfrak{p}(\Gamma_J) \overset{(5.2)}{\subseteq} \mathfrak{p}(\Gamma_{\mathcal{L}}) \overset{(5.3)}{=} P_\Gamma(\mathcal{L})$. Thus S_Γ and P_Γ are balanced (see (3.5)).

(iii) Let Γ be upper stable. If Γ is direct, $\Gamma_{\mathcal{L}/I_{\mathcal{L}}} = \{\{0\}\}$ for all $\mathcal{L} \in \mathfrak{L}$, where $I_{\mathcal{L}} = S_\Gamma(\mathcal{L})$. Hence $S_\Gamma(\mathcal{L}/S_\Gamma(\mathcal{L})) = S_\Gamma(\mathcal{L}/I_{\mathcal{L}}) \overset{(5.3)}{=} \mathfrak{s}(\Gamma_{\mathcal{L}/I_{\mathcal{L}}}) = \{0\}$.

If Γ is inverse, $p(\Gamma_{\mathcal{L}/I_{\mathcal{L}}}) = \{0\}$ for all $\mathcal{L} \in \mathfrak{L}$, where $I_{\mathcal{L}} = P_\Gamma(\mathcal{L})$. Hence $P_\Gamma(\mathcal{L}/P_\Gamma(\mathcal{L})) = P_\Gamma(\mathcal{L}/I_{\mathcal{L}}) \overset{(5.3)}{=} \mathfrak{p}(\Gamma_{\mathcal{L}/I_{\mathcal{L}}}) = \{0\}$. Thus (see (3.4)) S_Γ and P_Γ are upper stable. $\square$

Now we characterize S_Γ^*-radical and P_Γ°-semisimple Lie algebras via multifunctions Γ.

Theorem 5.6. *Let Γ be a subspace-multifunction on $\mathfrak{L}$.*

(i) *If Γ is direct, then the following are equivalent for each $\mathcal{L} \in \mathfrak{L}$.*
  1) *$\Gamma_{\mathcal{L}/I}$ is non-empty and $\Gamma_{\mathcal{L}/I} \neq \{\{0\}\}$ for each $I \lhd \mathcal{L}$, $I \neq \mathcal{L}$.*
  2) *$\Gamma_{\mathcal{L}/I}$ is non-empty and $\Gamma_{\mathcal{L}/I} \neq \{\{0\}\}$ for each $I \lhd^{\mathrm{ch}} \mathcal{L}$, $I \neq \mathcal{L}$.*
  3) *$\mathcal{L}$ is S_Γ^*-radical.*

(ii) *Let Γ be inverse and the preradical P_Γ balanced. The following are equivalent, for $\mathcal{L} \in \mathfrak{L}$.*
  1) *Γ_I is a non-empty family of proper subspaces for each $\{0\} \neq I \lhd \mathcal{L}$.*
  2) *Γ_I is a non-empty family of proper subspaces for each $\{0\} \neq I \lhd^{\mathrm{ch}} \mathcal{L}$.*
  3) *$\mathcal{L}$ is P_Γ°-semisimple.*

Proof. (i) 1) $\Longrightarrow$ 2) is evident. 2) $\Longrightarrow$ 3). Set $R = S_\Gamma$. By Proposition 5.3(i) and Theorem 4.5, R^* is an upper stable preradical and $R \leq R^*$. Set $I = R^*(\mathcal{L})$. Then I is a characteristic Lie ideal of $\mathcal{L}$ and $R(\mathcal{L}/I) \subseteq R^*(\mathcal{L}/I)$. If $\mathcal{L}$ is not R^*-radical then $I \neq \mathcal{L}$. As $\Gamma_{\mathcal{L}/I}$ is non-empty and $\Gamma_{\mathcal{L}/I} \neq \{\{0\}\}$, we have $R(\mathcal{L}/I) = S_\Gamma(\mathcal{L}/I) = \mathfrak{s}(\Gamma_{\mathcal{L}/I}) \neq \{0\}$. Hence $\{0\} \neq R(\mathcal{L}/I) \subseteq R^*(\mathcal{L}/I) = R^*(\mathcal{L}/R(\mathcal{L})) \overset{(3.4)}{=} \{0\}$. This contradiction implies that $R^*(\mathcal{L}) = \mathcal{L}$. Thus (see (3.7)) $\mathcal{L}$ is R-radical.

3) $\Longrightarrow$ 1). Let $\mathcal{L}$ be R^*-radical and let $I \lhd \mathcal{L}$, $I \neq \mathcal{L}$. By Lemma 3.6(i), $R^*(\mathcal{L}/I) = \mathcal{L}/I \neq \{0\}$. Hence it follows from Theorem 4.5 that $R(\mathcal{L}/I) = S_\Gamma(\mathcal{L}/I) \neq 0$. This is only possible when $\Gamma_{\mathcal{L}/I}$ is a non-empty family with non-zero subspaces.

(ii) 1) $\Longrightarrow$ 2) is evident. 2) $\Longrightarrow$ 3). Set $R = P_\Gamma$. By Proposition 5.3(ii) and Theorem 4.2, R° is an under radical. Let $\mathcal{L}$ be not R°-semisimple. By Lemma 2.2, $I = R^\circ(\mathcal{L}) \neq \{0\}$ is a characteristic Lie ideal of $\mathcal{L}$. Hence Γ_I is a non-empty family of proper subspaces of I. Then $R(R^\circ(\mathcal{L})) = P_\Gamma(I) = \mathfrak{p}(\Gamma_I) \subsetneqq I$. By Theorem 4.2, $R^\circ \leq R$. Therefore, as R° is lower stable, $R^\circ(\mathcal{L}) \overset{(3.3)}{=} R^\circ(R^\circ(\mathcal{L})) \subseteq R(R^\circ(\mathcal{L})) \subsetneqq R^\circ(\mathcal{L})$, a contradiction. Thus $\mathcal{L}$ is R°-semisimple.

3) $\Longrightarrow$ 1). Let $\mathcal{L}$ be R°-semisimple (that is, $R^\circ(\mathcal{L}) = \{0\}$) and let $\{0\} \neq I \lhd \mathcal{L}$. As R° is balanced, we have from Lemma 3.6(ii) that $R^\circ(I) = \{0\}$. If $R(I) = I$, it follows from (4.5) that $R^\circ(I) = I$, a contradiction. Thus $R(I) = P_\Gamma(I) = \mathfrak{p}(\Gamma_I) \subsetneqq I$. This is only possible when Γ_I is a non-empty family of proper subspaces of I. $\square$

Let Γ be a Lie subalgebra-multifunction on $\mathfrak{L}$, that is, each family $\Gamma_\mathcal{L}$, $\mathcal{L} \in \mathfrak{L}$, consists of closed Lie subalgebras of $\mathcal{L}$. If R is a preradical, then $R(\Gamma)$ is also a Lie subalgebra-multifunction on $\mathfrak{L}$, where each $R(\Gamma)_\mathcal{L} = R(\Gamma_\mathcal{L}) = \{R(L)\colon L \in \Gamma_\mathcal{L}\}$ is a family of closed Lie subalgebras of $\mathcal{L}$.

Proposition 5.7. *Let Γ be a Lie subalgebra-multifunction on $\mathfrak{L}$ and R be a preradical on $\overline{\mathbf{L}}$. If Γ is strictly direct then the multifunction $R(\Gamma)$ is direct. If R, in addition, is balanced and $\Gamma_J \subseteq \Gamma_\mathcal{L} \cap J$ for all $J \lhd \mathcal{L} \in \mathfrak{L}$ (see (5.4)), then $R(\Gamma)$ is balanced.*

Proof. Let $f\colon \mathcal{L} \longrightarrow \mathcal{M}$ be a homomorphism with dense range. As Γ is strictly direct, $M := \overline{f(L)} \in \Gamma_\mathcal{M}$, for each $L \in \Gamma_\mathcal{L}$, and $f|_L\colon L \longrightarrow M$ is a homomorphism with dense range. As R is a preradical on $\overline{\mathbf{L}}$, we have $f(R(L)) \subseteq R(M)$. Hence $f(R(\Gamma_\mathcal{L})) \overset{\rightarrow}{\subset} R(\Gamma_\mathcal{M})$.

Let $J \lhd \mathcal{L}$. Then, for each $L \in \Gamma_\mathcal{L}$, we have $(L \cap J) \lhd L$. If R is balanced then $R(L \cap J) \subseteq R(L)$. Hence if $\Gamma_J \subseteq \Gamma_\mathcal{L} \cap J$ then $R(\Gamma_J) \overset{\rightarrow}{\subset} R(\Gamma_\mathcal{L})$. $\square$

It follows from Proposition 5.3 and Theorem 5.5(ii) that in the conditions of Proposition 5.7 $S_{R(\Gamma)}$ is a preradical and a balanced preradical, respectively.

6. Examples of multifunctions and radicals

In the first subsection we consider some preliminary results about chains of closed subspaces which we will later apply to describe examples of multifunctions and radicals.

6.1. **Finite-gap families of subspaces.** In the following lemma we gather several elementary results on subspaces of finite codimension in a normed space.

Lemma 6.1. *Let Z be a subspace of a normed space X and let Y be a closed subspace of finite codimension in X. Then the subspace $Y + Z$ is closed in X, $Y \cap Z$ is a closed subspace of finite codimension in Z and* $\dim(Z/(Y \cap Z)) = \dim((Y + Z)/Y)$.

Proof. Let $q\colon X \longrightarrow X/Y$ be the quotient map. As X/Y is finite dimensional, $q(Y + Z)$ is closed in X/Y, whence $Y + Z = q^{-1}(q(Y + Z))$ is closed in X. It is clear that $Y \cap Z$ is closed in Z. As $(Y + Z)/Y$ and $Z/(Y \cap Z)$ are isomorphic in the pure algebraic sense, their dimensions coincide, whence $Y \cap Z$ has finite codimension in Z. $\square$

From now on X denotes a Banach space and G a family of closed subspaces of X. For $Y, Z \in G$ with $Y \subsetneqq Z$, the set $[Y, Z]_G = \{W \in G\colon Y \subseteq W \subseteq Z\}$ is called an *interval* of G. If $[Y, Z]_G = \{Y, Z\}$, the pair (Y, Z) is called a *gap*. Recall (see (4.1)) that

$$\mathfrak{p}(G) = X \text{ and } \mathfrak{s}(G) = \{0\}, \text{ if } G = \varnothing; \text{ otherwise } \mathfrak{p}(G) = \bigcap_{Y \in G} Y \text{ and } \mathfrak{s}(G) = \overline{\sum_{Y \in G} Y}.$$

For a family G of closed subspaces of X, define its $\mathfrak{p}$-*completion* and $\mathfrak{s}$-*completion* as follows:

$$G^\mathfrak{p} = \{\mathfrak{p}(G')\colon \varnothing \neq G' \subseteq G\} \cup \{\mathfrak{s}(G)\} \text{ and } G^\mathfrak{s} = \{\mathfrak{s}(G')\colon \varnothing \neq G' \subseteq G\} \cup \{\mathfrak{p}(G)\}.$$

We add $\mathfrak{s}(G)$ to $G^{\mathfrak{p}}$ and $\mathfrak{p}(G)$ to $G^{\mathfrak{s}}$ for technical convenience. We say that 1) G is $\mathfrak{p}$-*complete* if $G = G^{\mathfrak{p}}$; 2) G is $\mathfrak{s}$-*complete* if $G = G^{\mathfrak{s}}$, and 3) G is *complete* if it is $\mathfrak{p}$-complete and $\mathfrak{s}$-complete. Clearly, $G \subseteq G^{\mathfrak{p}} \cap G^{\mathfrak{s}}$.

Definition 6.2. *A family G of closed subspaces of X is called*

(i) *a* lower finite-gap family *if, for each $Z \neq \mathfrak{p}(G)$ in G, there is $Y \in G$ such that $Y \subset Z$ and $0 < \dim(Z/Y) < \infty$.*

(ii) *an* upper finite-gap family *if, for each $Z \neq \mathfrak{s}(G)$ in G, there is $Y \in G$ such that $Z \subset Y$ and $0 < \dim(Y/Z) < \infty$.*

Note that a lower finite-gap family may have infinite gaps. Let X be a Hilbert space with a basis $\{e_n\}_{n=1}^{\infty}$. The family G of subspaces $L_k = \{e_n\}_{n=k}^{\infty}$, $M_k = \{e_{2n-1}\}_{n=k}^{\infty}$, for all $1 \leq k < \infty$, and $\{0\}$ is a $\mathfrak{p}$-complete, lower finite-gap family. However, $[M_1, L_1]_G$ is an infinite gap.

The next lemma provides us with numerous examples of lower and upper finite-gap families.

Lemma 6.3. *Let G be a family of closed subspaces of X.*

(i) *If G consists of subspaces of finite codimension then $G^{\mathfrak{p}}$ is a lower finite-gap family.*

(ii) *If G consists of subspaces of finite dimension then $G^{\mathfrak{s}}$ is an upper finite-gap family.*

Proof. (i) Let $\mathfrak{p}(G) \neq Z \in G^{\mathfrak{p}}$. Then $Z = \mathfrak{p}(G')$ for some $\varnothing \neq G' \subsetneqq G$. Set $G'' = G \diagdown G'$. Then

$$\mathfrak{p}(G) = \mathfrak{p}\left(G'\right) \cap \mathfrak{p}\left(G''\right) = Z \cap \mathfrak{p}\left(G''\right) = \cap_{Y \in G''}(Z \cap Y).$$

As $Z \neq \mathfrak{p}(G)$, there is a subspace $Y \in G''$ such that $Z \cap Y \neq Z$. Then $Z \cap Y \in G^{\mathfrak{p}}$ and, by Lemma 6.1, $Z \cap Y$ has finite codimension in Z. Thus $0 < \dim(Z/(Z \cap Y)) < \infty$.

(ii) Let $\mathfrak{s}(G) \neq Z \in G^{\mathfrak{s}}$. Then $Z = \mathfrak{s}(G')$ for some $\varnothing \neq G' \subsetneqq G$. Set $G'' = G \diagdown G'$. Then

$$\mathfrak{s}(G) = \operatorname{span}\left(\mathfrak{s}\left(G'\right) + \cup_{Y \in G''} Y\right) = \operatorname{span}\left(\cup_{Y \in G''}(Z + Y)\right).$$

As $Z \neq \mathfrak{s}(G)$, there is a subspace $Y \in G''$ such that $Z + Y \neq Z$. By Lemma 6.1, $Z + Y$ is closed. Hence $Z + Y \in G^{\mathfrak{s}}$ and $0 < \dim((Z + Y)/Z) < \infty$. ☐

Remark 6.4. In this paper we consider $\mathfrak{p}$-complete lower finite-gap families. Similar results hold for $\mathfrak{s}$-complete upper finite-gap families.

A subfamily C of G is a *chain* if every two subspaces in C are comparable, that is, the order defined by inclusion is linear on C. A chain C is *maximal* if G has no other larger chain.

Lemma 6.5. *Let G be a $\mathfrak{p}$-complete family of closed subspaces in X. Then*

(i) *For each chain C_0 in G, there is a maximal $\mathfrak{p}$-complete chain C_m in G containing C_0 with $\mathfrak{p}(C_m) = \mathfrak{p}(C_0)$ and $\mathfrak{s}(C_m) = \mathfrak{s}(C_0)$.*

(ii) *Let C be a $\mathfrak{p}$-complete, lower finite-gap chain. Then*

  a) *C is complete and is a complete strictly decreasing transfinite sequence of closed subspaces;*

  b) *each chain in $[\mathfrak{p}(C), \mathfrak{s}(C)]_G$ larger than C is also a lower finite-gap chain;*

  c) *$[\mathfrak{p}(C), \mathfrak{s}(C)]_G$ has a maximal, complete lower finite-gap chain containing C.*

(iii) *If a lower finite-gap chain C is maximal in the interval $[\mathfrak{p}(C), \mathfrak{s}(C)]_G$, then C is complete.*

(iv) *Let C_0 be a $\mathfrak{p}$-complete, lower finite-gap chain with $\mathfrak{s}(C_0) = \mathfrak{s}(G)$. Then G has a maximal, lower finite-gap chain C containing C_0. If G is a lower finite-gap family, then $\mathfrak{p}(C) = \mathfrak{p}(G)$.*

Proof. (i) As G is $\mathfrak{p}$-complete, $C_0^{\mathfrak{p}}$ is a $\mathfrak{p}$-complete chain in G, $\mathfrak{p}(C_0^{\mathfrak{p}}) = \mathfrak{p}(C_0)$ and $\mathfrak{s}(C_0^{\mathfrak{p}}) = \mathfrak{s}(C_0)$. The set $\mathcal{G}$ of all $\mathfrak{p}$-complete chains C in G containing $C_0^{\mathfrak{p}}$, with $\mathfrak{p}(C) = \mathfrak{p}(C_0)$ and $\mathfrak{s}(C) = \mathfrak{s}(C_0)$, is partially ordered by inclusion. Let $\{C_\lambda\}_{\lambda \in \Lambda}$ be a linearly ordered subset of $\mathcal{G}$. Then $C' = (\cup_{\lambda \in \Lambda} C_\lambda)^{\mathfrak{p}} \in \mathcal{G}$. Hence $\mathcal{G}$ is inductive. By Zorn's Lemma, $\mathcal{G}$ has a maximal element C_m.

(ii) a) We only need to show that C is $\mathfrak{s}$-complete. Let $C_0 \neq \varnothing$ be a subset of C. If $C_0 = \{\mathfrak{p}(C_0)\}$ then $\mathfrak{s}(C_0) = \mathfrak{p}(C_0) \in C_0$. If $C_0 \neq \{\mathfrak{p}(C_0)\}$, set $C_1 = \{Y \in C\colon\ Z \subseteq Y$ for all $Z \in C_0\}$. As $\mathfrak{s}(C) \in C_1$, C_1 is not empty. It follows that $Z \subseteq \mathfrak{p}(C_1) \in C_1$ for each $Z \in C_0$.

Assume that $\mathfrak{p}(C_1) \notin C_0$. As C is a lower finite-gap chain, there is $Y_0 \in C$ such that $[Y_0, \mathfrak{p}(C_1)]_C$ is a gap. Hence $Y_0 \notin C_1$ and $Y_0 \subsetneqq Z_0$ for some $Z_0 \in C_0$, otherwise $Y_0 \in C_1$. Thus $Y_0 \subsetneqq Z_0 \subsetneqq \mathfrak{p}(C_1)$, so that $[Y_0, \mathfrak{p}(C_1)]_C$ is not a gap. This contradiction shows that $\mathfrak{p}(C_1) \in C_0$. Hence $\mathfrak{s}(C_0) = \mathfrak{p}(C_1)$.

We also proved that C is completely ordered by $\supseteq$. So it is anti-isomorphic to an interval $\lfloor 0, \beta \rfloor$ of transfinite numbers. Thus subspaces in C are indexed by transfinite numbers and C is a strictly decreasing transfinite sequence $\{Y_\alpha\}_{\alpha \le \beta}$ of closed subspaces ('strictly' means that $Y_{\alpha'} \neq Y_\alpha$ if $\alpha' \neq \alpha$).

b) Let $C \subset C_1 \subseteq [\mathfrak{p}(C), \mathfrak{s}(C)]_G$. Let $Y \in C_1 \diagdown C$. Then $\mathfrak{p}(C) \subsetneqq Y \subsetneqq \mathfrak{s}(C)$. The chain $C' = \{Z \in C\colon\ Y \subseteq Z\}$ is not empty, as $\mathfrak{s}(C) \in C'$, and $\mathfrak{p}(C') \in C'$, as C is $\mathfrak{p}$-complete. As C is a lower finite-gap chain and $\mathfrak{p}(C') \neq \mathfrak{p}(C)$, there is $Y_0 \in C$ such that $[Y_0, \mathfrak{p}(C')]_C$ is a finite gap. Hence $Y_0 \notin C'$, so that $Y_0 \subsetneqq Y \subsetneqq \mathfrak{p}(C')$. Thus $0 < \dim Y/Y_0 < \infty$. This implies that C_1 is a lower finite-gap chain.

c) By (i), $[\mathfrak{p}(C), \mathfrak{s}(C)]_G$ has a maximal $\mathfrak{p}$-complete chain containing C. By (ii) a) and b), it is a complete, lower finite-gap chain.

(iii) As G is $\mathfrak{p}$-complete, $C^{\mathfrak{p}}$ is a chain in $[\mathfrak{p}(C), \mathfrak{s}(C)]_G$ larger than C. As C is maximal, $C = C^{\mathfrak{p}}$. By (ii) a), C is complete.

(iv) As G is $\mathfrak{p}$-complete, $\mathfrak{p}(G), \mathfrak{s}(G) \in G$. Consider the set $\mathcal{G}$ of all lower finite-gap chains C in G containing C_0 and maximal in the interval $[\mathfrak{p}(C), \mathfrak{s}(G)]_G$. By (ii) c), $\mathcal{G}$ is not empty. It is partially ordered by inclusion. Let $\{C_\lambda\}_{\lambda \in \Lambda}$ be a linearly ordered subset of $\mathcal{G}$. By (iii), $\mathfrak{p}(C_\lambda) \in C_\lambda$. Set $Y_\lambda = \mathfrak{p}(C_\lambda)$, $Y_\Lambda = \mathfrak{p}\{Y_\lambda\colon\ \lambda \in \Lambda\}$ and

$$C_\Lambda = \left(\underset{\lambda \in \Lambda}{\cup} C_\lambda \right) \cup Y_\Lambda.$$

Then C_Λ is a chain, $C_0 \in C_\Lambda$ and $\mathfrak{p}(C_\Lambda) = Y_\Lambda \in C_\Lambda$. Each $V \in C_\Lambda$, $V \neq Y_\Lambda$, lies in some C_λ. If $Y_\lambda \neq V \in C_\lambda$, there is $W \in C_\lambda$ such that $W \subset V$ and $0 < \dim(V/W) < \infty$. If $V = Y_\lambda \neq Y_\Lambda$, there is $\mu \in \Lambda$ such that $V \in C_\lambda \subsetneqq C_\mu$ and $V \neq Y_\mu$. Hence, as in the previous case, there is $W \in C_\mu$ such that $W \subset V$ and $0 < \dim(V/W) < \infty$. Thus C_Λ is a lower finite-gap chain.

Let us show that C_Λ is a maximal chain in $[Y_\Lambda, \mathfrak{s}(G)]_G$. If not, then there is $V \notin C_\Lambda$, $V \in [Y_\Lambda, \mathfrak{s}(G)]_G$ such that $C_\Lambda \cup V$ is a chain. As $Y_\Lambda \subsetneqq V$, there is $\lambda \in \Lambda$ such that $Y_\lambda \subsetneqq V$. Then $C_\lambda \cup V$ is a chain in $[Y_\lambda, \mathfrak{s}(G)]_G$ larger than C_λ. This contradiction shows that the chain C_Λ is maximal in $[Y_\Lambda, \mathfrak{s}(G)]_G$.

Clearly, $C_\Lambda \le C'$ for each majorant C' of $\{C_\lambda\}_{\lambda \in \Lambda}$ in $\mathcal{G}$. Thus each linearly ordered subset of $\mathcal{G}$ has a supremum in $\mathcal{G}$. By Zorn's Lemma, $\mathcal{G}$ has a maximal element C which is a lower finite-gap chain containing C_0 and maximal in the interval $[\mathfrak{p}(C), \mathfrak{s}(G)]_G$.

Let G be a lower finite-gap family. If $\mathfrak{p}(C) \neq \mathfrak{p}(G)$ then, as G is a lower finite-gap family, there is $W \in G$ such that $W \subsetneqq \mathfrak{p}(C)$ and $\dim(\mathfrak{p}(C)/W) < \infty$. Choose a finite maximal chain C_1 in $[W, \mathfrak{p}(C)]_G$. Then the chain $C \cup C_1$ belongs to $\mathcal{G}$ and larger than C — a contradiction. Thus $\mathfrak{p}(C) = \mathfrak{p}(G)$. $\square$

Theorem 6.6. *A $\mathfrak{p}$-complete family G of closed subspaces of X is a lower finite-gap family if and only if there is a maximal, lower finite-gap chain C in G with $\mathfrak{p}(C) = \mathfrak{p}(G)$ and $\mathfrak{s}(C) = \mathfrak{s}(G)$.*

Proof. $\Longrightarrow$ follows from Lemma 6.5.

$\Longleftarrow$ Let C satisfy the conditions of the theorem and let $\mathfrak{p}(G) \neq Z \in G$. We need to show that there is $Y_1 \subset Z$ such that $0 < \dim(Z/Y_1) < \infty$. It is evident if $Z \in C$. Let $Z \notin C$. The chain $C_1 = \{Y \in C\colon\ Z \subseteq Y\}$ is not empty because at least $\mathfrak{s}(G) \in C_1$. Clearly, $Z \subseteq \mathfrak{p}(C_1)$. As $\mathfrak{p}(C_1) \neq \mathfrak{p}(G) = \mathfrak{p}(C)$, there is a subspace Y in C with $0 < \dim(\mathfrak{p}(C_1)/Y) < \infty$. Hence, by Lemma 6.1, $Y_1 = Y \cap Z$ has finite codimension in Z. Since $Y \notin C_1$, this codimension is non-zero. As G is $\mathfrak{p}$-complete, $Y_1 = \mathfrak{p}(\{Y, Z\}) \in G$. Thus G is a lower finite-gap family. $\square$

We will show now that complete, lower finite-gap families of subspaces of X induce complete, lower finite-gap families of subspaces on closed subspaces of X.

Corollary 6.7. *Let G be a $\mathfrak{p}$-complete, lower finite-gap family of closed subspaces of X. For any closed subspace W of X, $G \cap W := \{Y \cap W : Y \in G\}$ is a $\mathfrak{p}$-complete, lower finite-gap family.*

Proof. We have $\mathfrak{p}(G\cap W) = \mathfrak{p}(G)\cap W$. If $Y \cap W = p(G)\cap W$ for all $Y \in G$, then $G\cap W$ consists of one element and our corollary holds.

Let $Z \in G$ be such that $Z \cap W \neq \mathfrak{p}(G) \cap W$. Set $G_1 = \{T \in G\colon T \cap W = Z \cap W\}$. Then $\mathfrak{p}(G_1) \in G$ and $\mathfrak{p}(G_1) \cap W = Z \cap W \neq \mathfrak{p}(G) \cap W$. Hence $\mathfrak{p}(G) \subsetneqq \mathfrak{p}(G_1) \in G_1$. As G is a lower finite-gap family, there is $Y \in G$ such that $Y \subset \mathfrak{p}(G_1)$ and $0 < \dim(\mathfrak{p}(G_1)/Y) < \infty$. Replacing in Lemma 6.1 X by $\mathfrak{p}(G_1)$ and Z by $\mathfrak{p}(G_1) \cap W$, we get that $Y \cap (\mathfrak{p}(G_1) \cap W) = Y \cap W$ has finite codimension in $\mathfrak{p}(G_1) \cap W = Z \cap W$. As $Y \in G\diagdown G_1$, we have $Y \cap W \subsetneqq Z \cap W$. Thus $0 < \dim (Z \cap W) / (Y \cap W) < \infty$. This means that $G \cap W$ is a lower finite-gap family.

As the family G is $\mathfrak{p}$-complete, it follows that the family $G \cap W$ is also $\mathfrak{p}$-complete. □

Note that if G is a lower finite-gap family of closed subspaces, $G^{\mathfrak{p}}$ is not necessarily a lower finite-gap family. For example, if G is the family of all closed subspaces of finite codimension in X, then $G^{\mathfrak{p}}$ is the family of all closed subspaces of X.

Proposition 6.8. *Let $G = \cup_{\lambda\in\Lambda}G_\lambda$ where each G_λ is a $\mathfrak{p}$-complete, lower finite-gap family of closed subspaces of X. If $X \in G_\lambda$, for all $\lambda \in \Lambda$, then $G^{\mathfrak{p}}$ is a lower finite-gap family.*

Proof. Let $Y \in G^{\mathfrak{p}}$ and $\mathfrak{p}(G) \subsetneqq Y$. Then $Y = \mathfrak{p}(G')$ for some $\varnothing \neq G' \subsetneqq G$. Set $\Gamma_\lambda = G_\lambda \diagdown (G' \cap G_\lambda)$ for each $\lambda \in \Lambda$. Then $G\diagdown G' = \cup_\lambda \Gamma_\lambda$ and

$$\mathfrak{p}(G) = \mathfrak{p}\left(G\diagdown G'\right) \cap \mathfrak{p}\left(G'\right) = (\cap_{\lambda\in\Lambda}\mathfrak{p}(\Gamma_\lambda)) \cap Y = \cap_{\lambda\in\Lambda}(\mathfrak{p}(\Gamma_\lambda) \cap Y).$$

Hence $\mathfrak{p}(\Gamma_\lambda) \cap Y \neq Y$, for some λ. Thus

$$\mathfrak{p}(G_\lambda \cap Y) = \mathfrak{p}(G_\lambda) \cap Y = \mathfrak{p}(\Gamma_\lambda) \cap \mathfrak{p}(G' \cap G_\lambda) \cap Y = \mathfrak{p}(\Gamma_\lambda) \cap Y \neq Y.$$

By Corollary 6.7, $G_\lambda \cap Y$ is a lower finite-gap family and $Y = X \cap Y \in G_\lambda \cap Y$. Hence there is $Z \in G_\lambda$ such that $0 < \dim (Y/(Z \cap Y)) < \infty$. As $Z \cap Y \in G^{\mathfrak{p}}$, then $G^{\mathfrak{p}}$ is a lower finite-gap family. □

We need now the following auxiliary result.

Lemma 6.9. *Let $\{X_\lambda : \lambda \in \Lambda\}$ be a family of closed subspaces of a separable Banach space X.*

(i) *If $\cap_{\lambda\in\Lambda}X_\lambda = \{0\}$ then there is a sequence $\{\lambda_n \in \Lambda : n \in \mathbb{N}\}$ such that $\cap_{n=1}^{\infty}X_{\lambda_n} = \{0\}$.*

(ii) *If $\overline{\sum_{\lambda\in\Lambda} X_\lambda} = X$ then there is a sequence $\{\lambda_n \in \Lambda : n \in \mathbb{N}\}$ such that $\overline{\sum_{n=1}^{\infty} X_{\lambda_n}} = X$.*

Proof. (i) For each $\lambda \in \Lambda$, set $W_\lambda = \{f \in X^*\colon \|f\| \leq 1, f(x) = 0 \text{ for all } x \in X_\lambda\}$. Then $W = \cup_{\lambda\in\Lambda}W_\lambda$ is a subset of the unit ball $\mathbf{B}$ of X^*. As X is separable, $\mathbf{B}$ is a separable metric space in the weak* topology (see [Sch, Section 4.1.7]). Hence W has a weak* dense sequence $\{f_n\colon n \in \mathbb{N}\}$. It follows that $\cap_n \ker(f_n) = \cap_{f\in W} \ker(f) = \cap_\lambda X_\lambda = \{0\}$. Choosing an index λ_n such that $f_n \in W_{\lambda_n}$ for each n, we get $\cap_{n=1}^{\infty}X_{\lambda_n} \subseteq \cap_n \ker(f_n) = \{0\}$.

(ii) Set $E = \cup_{\lambda\in\Lambda}X_\lambda$. Then X is the closed linear span of E. As E is separable, it has a dense sequence $\{x_n\colon n \in \mathbb{N}\}$. Choosing λ_n such that $x_n \in X_{\lambda_n}$ for each n, we get $\overline{\sum_{n=1}^{\infty} X_{\lambda_n}} = X$. □

Theorem 6.10. *Let G be a family of closed subspaces of finite codimension in X. Then there is a maximal, lower finite-gap chain C of subspaces in $G^{\mathfrak{p}}$ with $\mathfrak{p}(C) = \mathfrak{p}(G)$ and $\mathfrak{s}(C) = \mathfrak{s}(G)$.*

Suppose that the quotient space $\mathfrak{s}(G)/\mathfrak{p}(G)$ is separable and infinite-dimensional. Then there are subspaces $\{V_k\}_{k=1}^{\infty}$ in G such that their finite intersections $Y_n = \cap_{k=1}^{n}V_k$ together with $\mathfrak{s}(G)$ form a decreasing complete, lower finite-gap chain between $\mathfrak{s}(G)$ and $\mathfrak{p}(G) = \cap_{n=1}^{\infty}Y_n$.

Proof. The existence of the chain C follows from Lemmas 6.3 and 6.5.

Let $\mathfrak{s}(G)/\mathfrak{p}(G)$ be separable and infinite-dimensional. First assume that $\mathfrak{s}(G) = X$ and $\mathfrak{p}(G) = \{0\}$. By Lemma 6.9, there is a sequence $\{W_i\}$ in G such that $\cap_{i=1}^{\infty}W_i = \{0\}$. Taking a subsequence, if necessary, one can assume that $W_{i_1} \neq W_{i_1} \cap W_{i_2} \neq \ldots \neq \cap_{k=1}^{n}W_{i_k} \neq \ldots$ and $\cap_{k=1}^{\infty}W_{i_k} = \cap_{i=1}^{\infty}W_i$. Setting $V_k = W_{i_k}$ for every k, we get the required sequence.

The general case is reduced to the above one if we take $\mathfrak{s}(G)/\mathfrak{p}(G)$ instead of X. By the above, we can find a required sequence $\{V_i'\}$ for the family $G' = \{V/\mathfrak{p}(G)\colon V \in G\}$. Taking the preimages V_i of V_i' in $\mathfrak{s}(G)$, we obtain the required sequence. □

Let G be a $\mathfrak{p}$-complete family of closed subspaces of X and $\mathfrak{s}(G) = X$. By Lemma 6.5, G has a maximal, lower finite-gap chain C with $\mathfrak{s}(C) = X$. Let G_{f} be the subset of G that consists of all $Y \in G$ such that there is a $\mathfrak{p}$-complete, lower finite-gap chain C_Y with $\mathfrak{s}(C_Y) = X$ and $\mathfrak{p}(C_Y) = Y$. Set

$$\Delta_G = \mathfrak{p}(G_{\mathrm{f}}). \tag{6.1}$$

Theorem 6.11. (i) *The subset G_{f} of G is a lower finite-gap family.*

(ii) $\Delta_G = \mathfrak{p}(C)$ *for each maximal, lower finite-gap chain C in G with $\mathfrak{s}(C) = X$.*

Proof. Let C, C' be maximal, lower finite-gap chains in G with $\mathfrak{s}(C) = \mathfrak{s}(C') = X$. Set $Y = \mathfrak{p}(C)$ and $Z = \mathfrak{p}(C')$. Then $Y, Z \in G$. By Corollary 6.7, $C \cap Z = \{U \cap Z \colon U \in C\}$ is a $\mathfrak{p}$-complete, lower finite-gap chain in Z. Suppose that $Z \nsubseteq Y$. Then $\mathfrak{s}(C \cap Z) = Z$, as $X \in C$, and $\mathfrak{p}(C \cap Z) = Y \cap Z \neq Z$. Hence $C' \cup (C \cap Z)$ is a $\mathfrak{p}$-complete, lower finite-gap family larger than C' — a contradiction. Hence $Z \subseteq Y$. Similarly, $Y \subseteq Z$. Thus $Y = Z$. This immediately implies (ii).

If $Y \in G_{\mathrm{f}}$ and $Y \neq \Delta_G$ then the chain C_Y is not maximal. By Lemma 6.5, there is a maximal, lower finite-gap chain C in G with $\mathfrak{s}(C) = X$ that contains C_Y. Hence there is a subspace $Z \in C$ such that $Z \subsetneqq Y$ and $\dim Y/Z < \infty$. As $Z \cup C_Y$ is a $\mathfrak{p}$-complete, lower finite-gap family, $Z \in G_{\mathrm{f}}$. This proves (i). $\square$

Note that G may have many different maximal, lower finite-gap chains starting at X. However, they all end at the same subspace Δ_G.

Let L be a Lie algebra of operators on a Banach space X. The set Lat L of all closed subspaces of X invariant for all operators in L is $\mathfrak{p}$-complete. Let $\mathrm{Lat}_{\mathrm{cf}}L = \{Y \in \mathrm{Lat}\ L : Y$ has finite codimension in $X\}$. Then Lemma 6.5 and Theorems 6.10 and 6.11 yield

Corollary 6.12. (i) *There is a subspace $\Delta_L \in$ Lat L such that $\mathfrak{p}(C) = \Delta_L$ for each maximal, lower finite-gap chain C of invariant subspaces of L with $s(C) = X$, and Δ_L has no invariant subspaces of finite codimension. If* Lat L *is a lower finite-gap family then $\Delta_L = \{0\}$.*

(ii) *If X is separable then there is a sequence $\{Y_n\}_{n=0}^{\infty}$ of subspaces in $\mathrm{Lat}_{\mathrm{cf}}L$ such that $Y_0 = X$, $Y_{n+1} \subset Y_n$ and $\cap_n Y_n = \mathfrak{p}(\mathrm{Lat}_{\mathrm{cf}}L)$.*

Definition 6.13. *Let G and G' be families of closed subspaces of X. Then G is called a* lower finite-gap family modulo G' *if, for each Z in G, $Z \neq \mathfrak{p}(G \cup G')$, there is $Y \in G \cup G'$ such that $Y \subset Z$ and $0 < \dim(Z/Y) < \infty$.*

Combining this definition and Definition 6.2, we obtain

Lemma 6.14. *Let G and G' be families of closed subspaces of X. If G is a lower finite-gap family modulo G' and G' is a lower finite-gap family, then $G \cup G'$ is a lower finite-gap family.*

6.2. **Preradicals corresponding to finite-dimensional Lie subalgebra-multifunctions.** For a Banach Lie algebra $\mathcal{L}$, the sequences $\{\mathcal{L}^{[n+1]}\}$ and $\{\mathcal{L}_{[n+1]}\}$ of closed characteristic Lie ideals

$$\mathcal{L}^{[1]} = \mathcal{L}_{[0]} = \mathcal{L},\ \mathcal{L}^{[n+1]} = \overline{[\mathcal{L}, \mathcal{L}^{[n]}]} \text{ and } \mathcal{L}_{[n+1]} = \overline{[\mathcal{L}_{[n]}, \mathcal{L}_{[n]}]},\ \text{for } n \in \mathbb{N}, \tag{6.2}$$

decrease; $\mathcal{L}$ is *nilpotent*, if $\mathcal{L}^{[n]} = \{0\}$ for some n, and *solvable* if $\mathcal{L}_{[n]} = \{0\}$ for some n.

Denote by $\mathfrak{L}^{\mathrm{f}}$ the class of all finite-dimensional Lie algebras and by $\mathbf{L}^{\mathrm{f}}$ the subcategory of $\mathbf{L}$ of all such algebras. As in (3.3)—(3.5) and Definition 3.4, we define lower stable, upper stable and balanced preradicals, under radicals, over radicals and radicals on $\mathbf{L}^{\mathrm{f}}$.

For $\mathcal{L} \in \mathfrak{L}^{\mathrm{f}}$, denote by rad$(\mathcal{L})$ its maximal solvable Lie ideal. The map rad: $\mathcal{L} \in \mathfrak{L}^{\mathrm{f}} \mapsto \mathrm{rad}(\mathcal{L})$ is a radical in $\mathbf{L}^{\mathrm{f}}$. A Lie algebra $\mathcal{L}$ is called *semisimple* if it is rad-semisimple; $\mathcal{L}$ is semisimple if and only if it is a direct sum of simple algebras. We preserve this terminology when dealing with finite-dimensional subalgebras of Banach Lie algebras.

Each $\mathcal{L} \in \mathfrak{L}^{\mathrm{f}}$ is the semidirect product (Levi-Maltsev decomposition) of a semisimple Lie subalgebra $N_{\mathcal{L}}$ (uniquely defined up to an inner automorphism) and the largest solvable Lie ideal rad$(\mathcal{L})$

$$\mathcal{L} = N_{\mathcal{L}} \oplus^{\mathrm{ad}|_{\mathrm{rad}(\mathcal{L})}} \mathrm{rad}(\mathcal{L}). \tag{6.3}$$

Recall that if Γ is a Lie subalgebra-multifunction or ideal-multifunction on $\mathfrak{L}$ then, for each $\mathcal{L}\in\mathfrak{L}$, $\Gamma_{\mathcal{L}}$ is a family of closed Lie subalgebras (ideals) of $\mathcal{L}$. In the rest of this subsection we will consider the following four Lie subalgebra-multifunctions Γ on $\mathfrak{L}$:

1) $\mathrm{A}^{\mathrm{sem}}$: each family $\mathrm{A}^{\mathrm{sem}}_{\mathcal{L}}$ consists of all finite-dimensional semisimple Lie subalgebras of $\mathcal{L}$.
2) $\mathrm{I}^{\mathrm{sem}}$: each family $\mathrm{I}^{\mathrm{sem}}_{\mathcal{L}}$ consists of all finite-dimensional semisimple Lie ideals of $\mathcal{L}$.
3) $\mathrm{I}^{\mathrm{sol}}$: each family $\mathrm{I}^{\mathrm{sol}}_{\mathcal{L}}$ consists of all finite-dimensional solvable Lie ideals of $\mathcal{L}$.
4) $\mathrm{I}^{\mathrm{fin}}$: each family $\mathrm{I}^{\mathrm{fin}}_{\mathcal{L}}$ consists of all finite-dimensional Lie ideals of $\mathcal{L}$.

We will study the corresponding radicals and describe their restrictions to $\mathbf{L}^{\mathrm{f}}$.

Proposition 6.15. (i) *The Lie subalgebra-multifunctions* Γ : $\mathrm{A}^{\mathrm{sem}}$, $\mathrm{I}^{\mathrm{sem}}$, $\mathrm{I}^{\mathrm{sol}}$, $\mathrm{I}^{\mathrm{fin}}$ *on* $\mathfrak{L}$ *are strictly direct and lower stable* (*see Definition* 5.4), *so that the corresponding maps* S_Γ *are lower stable preradicals.*

(ii) *The multifunctions* Γ : $\mathrm{A}^{\mathrm{sem}}$ *and* $\mathrm{I}^{\mathrm{sem}}$ *are balanced* (*see Definition* 5.4), *so that the corresponding maps* S_Γ *are under radicals and* $S_{\mathrm{I}^{\mathrm{sem}}}\leq S_{\mathrm{A}^{\mathrm{sem}}}$.

Proof. (i) Let $f:\mathcal{L}\longrightarrow\mathcal{M}$ be a morphism in $\overline{\mathbf{L}}$ and let L be a finite-dimensional Lie subalgebra of $\mathcal{L}$. Then $\overline{f(L)}=f(L)$ is a finite-dimensional Lie subalgebra of $\mathcal{M}$. If L is a Lie ideal of $\mathcal{L}$, then $f(L)$ is a Lie ideal of $f(\mathcal{L})$. As $f(\mathcal{L})$ is dense in $\mathcal{M}$, $f(L)$ is a Lie ideal of $\mathcal{M}$.

If L is solvable, $f(L)$ is solvable. If L is simple, $f(L)$ is either $\{0\}$ or simple. If L is semisimple, it is a finite direct sum of simple Lie algebras. Hence $f(L)$ is either $\{0\}$ or a semisimple Lie subalgebra of $\mathcal{M}$. This shows that all multifunctions are strictly direct. By Proposition 5.3, all S_Γ are preradicals.

Set $I_{\mathcal{L}}=S_\Gamma(\mathcal{L})=\mathfrak{s}(\Gamma_{\mathcal{L}})$. If $L\in\Gamma_{\mathcal{L}}$, it follows from (4.2) that $L\subseteq I_{\mathcal{L}}$. Hence $L\in\Gamma_{I_{\mathcal{L}}}$. Thus $\Gamma_{\mathcal{L}}\subseteq\Gamma_{I_{\mathcal{L}}}$, so that all multifunctions are lower stable (see Definition 5.4). By Theorem 5.5(i), all preradicals S_Γ are lower stable.

(ii) Let $I\lhd\mathcal{L}$. If $\Gamma=\mathrm{A}^{\mathrm{sem}}$ then $\Gamma_I\subseteq\Gamma_{\mathcal{L}}$. Let $\Gamma=\mathrm{I}^{\mathrm{sem}}$ and J be a semisimple Lie ideal of I. Then $J=[J,J]$. By Lemma 2.4(iii), J is a characteristic Lie ideal of I and, by Lemma 2.4(i), J is a semisimple Lie ideal of $\mathcal{L}$. Hence $\Gamma_I\subseteq\Gamma_{\mathcal{L}}$. Thus the multifunctions $\mathrm{A}^{\mathrm{sem}}$ and $\mathrm{I}^{\mathrm{sem}}$ are balanced. Hence, by Theorem 5.5(ii), all S_Γ are balanced. Thus they are under radicals. Clearly, $S_{\mathrm{I}^{\mathrm{sem}}}\leq S_{\mathrm{A}^{\mathrm{sem}}}$. $\square$

Combining this with Theorems 4.6 and 4.8 yields

Corollary 6.16. (i) *If* $\Gamma=\mathrm{A}^{\mathrm{sem}}$ *or* $\Gamma=\mathrm{I}^{\mathrm{sem}}$ *then the maps* S^*_Γ *are radicals.*

(ii) *If* $\Gamma=\mathrm{I}^{\mathrm{fin}}$ *or* $\Gamma=\mathrm{I}^{\mathrm{sol}}$ *then the maps* S^s_Γ *are under radicals and* $(S^s_\Gamma)^*$ *are radicals.*

Theorem 5.6(i) and Corollary 6.16(i) yield

Corollary 6.17. *A Banach Lie algebra* $\mathcal{L}$ *is* $S^*_{\mathrm{A}^{\mathrm{sem}}}$*-radical* (*respectively,* $S^*_{\mathrm{I}^{\mathrm{sem}}}$*-radical*) *if and only if, for each closed proper Lie ideal* I *of* $\mathcal{L}$, *the quotient* $\mathcal{L}/I$ *contains a finite-dimensional semisimple Lie subalgebra* (*respectively, ideal*).

Corollary 6.18. *If* $\mathcal{L}$ *is an* $S^*_{\mathrm{A}^{\mathrm{sem}}}$*-radical or an* $S^*_{\mathrm{I}^{\mathrm{sem}}}$*-radical, then* $\mathcal{L}=\overline{[\mathcal{L},\mathcal{L}]}$.

Proof. If $\overline{[\mathcal{L},\mathcal{L}]}\neq\mathcal{L}$ then $\mathcal{L}$ is not $S^*_{\mathrm{A}^{\mathrm{sem}}}$-radical (or $S^*_{\mathrm{I}^{\mathrm{sem}}}$-radical) because the algebra $\mathcal{L}/\overline{[\mathcal{L},\mathcal{L}]}$ cannot contain semisimple subalgebras (see Corollary 6.17). $\square$

Let $\mathcal{L}\in\mathfrak{L}^{\mathrm{f}}$. Then $\mathcal{L}_{[k+1]}=\mathcal{L}_{[k]}$ for some k. Set

$$\mathfrak{P}(\mathcal{L})=\cap\mathcal{L}_{[k]}. \tag{6.4}$$

The next theorem describes the restriction of the preradical $S_{\mathrm{A}^{\mathrm{sem}}}$ to $\mathbf{L}^{\mathrm{f}}$.

Theorem 6.19. (i) *The restriction of* $S_{\mathrm{A}^{\mathrm{sem}}}$ *to* $\mathbf{L}^{\mathrm{f}}$ *is a radical.*

(ii) *For each* $\mathcal{L}\in\mathfrak{L}^{\mathrm{f}}$, $S_{\mathrm{A}^{\mathrm{sem}}}(\mathcal{L})=\mathfrak{P}(\mathcal{L})$ *and it is the smallest characteristic Lie ideal of* $\mathcal{L}$ *that contains all Levi subalgebras* $N_{\mathcal{L}}$ (see (6.3)).

(iii) *A Lie algebra* $\mathcal{L}\in\mathfrak{L}^{\mathrm{f}}$ *is* $S_{\mathrm{A}^{\mathrm{sem}}}$*-semisimple if and only if* $\mathcal{L}$ *is solvable.*

Proof. (i) Set $\Gamma = \mathrm{A}^{\mathrm{sem}}$ and $I_{\mathcal{L}} = S_\Gamma(\mathcal{L}) = \mathfrak{s}(\Gamma_{\mathcal{L}})$. By Proposition 6.15, S_Γ is an under radical. If $S_\Gamma|\mathbf{L}^f$ is not a radical, $\mathfrak{s}(\Gamma_{\mathcal{L}/I_{\mathcal{L}}}) = S_\Gamma(\mathcal{L}/I_{\mathcal{L}}) \neq \{0\}$ for some $\mathcal{L} \in \mathfrak{L}^{\mathrm{f}}$. Hence $\mathcal{L}/I_{\mathcal{L}}$ contains a semisimple Lie subalgebra $M \neq \{0\}$. Let q: $\mathcal{L} \longrightarrow \mathcal{L}/I_{\mathcal{L}}$ be the quotient map and $L = q^{-1}(M)$. Let $L = N_L \dotplus \mathrm{rad}\,(L)$ be the Levi-Maltsev decomposition, where N_L is a semisimple Lie subalgebra of $\mathcal{L}$. As $N_L \subseteq I_{\mathcal{L}}$, we have that $M = q\,(L) = q\,(\mathrm{rad}\,(L))$ is solvable, a contradiction.

(ii) Let J be the minimal characteristic Lie ideal of $\mathcal{L}$ that contains some Levi subalgebra $N_{\mathcal{L}}$. As $N_{\mathcal{L}} \in \Gamma_{\mathcal{L}}$, we have $N_{\mathcal{L}} \subseteq \mathfrak{s}(\Gamma_{\mathcal{L}}) = I_{\mathcal{L}}$. As $I_{\mathcal{L}}$ is a characteristic Lie ideal, $J \subseteq I_{\mathcal{L}}$. Let M be a semisimple Lie subalgebra of $\mathcal{L}$. By [Bo, Corollary 1.6.8.1], there is x in $\mathcal{L}$ such that $\exp\,(\mathrm{ad}\,(x))\,(M) \subseteq N_{\mathcal{L}}$. As J is a characteristic Lie ideal of $\mathcal{L}$, $M \subseteq \exp\,(\mathrm{ad}\,(-x))\,(N_{\mathcal{L}}) \subseteq \exp\,(\mathrm{ad}\,(-x))\,J \subseteq J$. Hence all semisimple Lie subalgebras of $\mathcal{L}$ lie in J. Therefore $I_{\mathcal{L}} \subseteq J$. Thus $I_{\mathcal{L}} = J$.

As $N_{\mathcal{L}}$ is semisimple, $N_{\mathcal{L}} = [N_{\mathcal{L}}, N_{\mathcal{L}}]$, whence $N_{\mathcal{L}} \subseteq \mathcal{L}_{[2]}$. Similarly, $N_{\mathcal{L}} \subseteq \mathcal{L}_{[n]}$ for all n, so that $N_{\mathcal{L}} \subseteq \mathfrak{P}\,(\mathcal{L})$. As all $\mathcal{L}_{[n]}$ are characteristic Lie ideals of $\mathcal{L}$, $\mathfrak{P}\,(\mathcal{L})$ is a characteristic Lie ideal of $\mathcal{L}$ that contains $N_{\mathcal{L}}$. Hence $I_{\mathcal{L}} \subseteq \mathfrak{P}\,(\mathcal{L})$. As S_Γ is a radical, $S_\Gamma(\mathcal{L}/I_{\mathcal{L}}) = \{0\}$. Hence, by the definition of S_Γ, $\mathcal{L}/I_{\mathcal{L}}$ has no semisimple Lie subalgebras. Thus $\mathcal{L}/I_{\mathcal{L}}$ is solvable. Therefore $\mathfrak{P}\,(\mathcal{L}) = \mathcal{L}_{[k]} \subseteq I_{\mathcal{L}}$.

Part (iii) follows from the fact that $\{0\} = S_\Gamma\,(\mathcal{L}) = \mathcal{L}_{[k]}$ for some k. $\square$

Theorem 6.20. (i) *For each* $\mathcal{L} \in \mathfrak{L}^{\mathrm{f}}$, $S_{\mathrm{I}^{\mathrm{sem}}}\,(\mathcal{L})$ *is the largest semisimple Lie ideal of* $\mathcal{L}$.

(ii) *The restriction of* $S_{\mathrm{I}^{\mathrm{sem}}}$ *to* $\mathbf{L}^{\mathrm{f}}$ *is a hereditary radical* (*see* (3.6)).

Proof. (i) Set $\Gamma = I^{\mathrm{sem}}$. Let $\mathcal{L} \in \mathfrak{L}^{\mathrm{f}}$, let $I \lhd \mathcal{L}$ and $J \lhd \mathcal{L}$. If J is simple, then either $J = I$ or $J \cap I = \{0\}$. If $J \subseteq I$ and I is semisimple, then $I = I_1 \dotplus ... \dotplus I_n$, where all $\{I_i\}$ are simple Lie ideals of I, and J coincides with one of I_i. As all $I_i = [I_i, I_i]$, we have from Lemma 2.4(iii) that all I_i are simple Lie ideals of $\mathcal{L}$. Let $\{I_j\}_{j=1}^m$ be the set of *all* simple Lie ideals of $\mathcal{L}$. It follows from the discussion above that $K = I_1 \dotplus ... \dotplus I_m$ is the largest semisimple Lie ideal of $\mathcal{L}$ and it contains all semisimple Lie ideals of $\mathcal{L}$. Hence $S_\Gamma\,(\mathcal{L}) = \mathfrak{s}(\Gamma_{\mathcal{L}}) = K$.

(ii) As in Theorem 6.19(i), one can prove that $S_\Gamma|\mathbf{L}^{\mathrm{f}}$ is a radical. Let $I \lhd \mathcal{L}$. Then $S_\Gamma\,(I)$ is the largest semisimple Lie ideal of I. As $S_\Gamma\,(I)$ is a characteristic Lie ideal of I, by Lemma 2.4(i), $S_\Gamma\,(I)$ is the largest semisimple Lie ideal of $\mathcal{L}$ contained in I. Therefore $S_\Gamma\,(I) \subseteq S_\Gamma\,(\mathcal{L}) \cap I$. As $S_\Gamma\,(\mathcal{L}) \cap I$ is a Lie ideal of the semisimple Lie algebra $S_\Gamma\,(\mathcal{L})$, it is semisimple. Hence $S_\Gamma\,(\mathcal{L}) \cap I$ is a semisimple Lie ideal of I. By (i), $S_\Gamma\,(\mathcal{L}) \cap I \subseteq S_\Gamma\,(I)$. Thus $S_\Gamma\,(\mathcal{L}) \cap I = S_\Gamma\,(I)$, so that S_Γ is hereditary on $\mathbf{L}^{\mathrm{f}}$. $\square$

To see an example that distinguishes the radicals $S_{\mathrm{A}^{\mathrm{sem}}}|\mathbf{L}^{\mathrm{f}}$ and $S_{\mathrm{I}^{\mathrm{sem}}}|\mathbf{L}^{\mathrm{f}}$, consider the semidirect product $\mathcal{L} = sl(X) \oplus^{\mathrm{id}} X$, where X is a finite-dimensional space and $sl(X)$ the Lie algebra of all operators on X with zero trace. It has no semisimple ideals, so that $\mathrm{I}_{\mathcal{L}}^{\mathrm{sem}} = \varnothing$ and $S_{\mathrm{I}^{\mathrm{sem}}}(\mathcal{L}) = \{0\}$, while $S_{\mathrm{A}^{\mathrm{sem}}}(\mathcal{L}) = \mathcal{L}$ because the only ideal that contains the semisimple Levi subalgebra $sl(X) \oplus^{\mathrm{id}} \{0\}$ is $\mathcal{L}$ itself.

We call the restriction of $S_{\mathrm{A}^{\mathrm{sem}}}$ to $\mathbf{L}^{\mathrm{f}}$ the *Levi radical* and denote it by R_{Levi}:

$$R_{\mathrm{Levi}} = S_{\mathrm{A}^{\mathrm{sem}}}|\mathbf{L}^{\mathrm{f}}.$$

It is *not hereditary.* Indeed, let $L \in \mathbf{L}^{\mathrm{f}}$ be semisimple and let π be an irreducible representation of L on a finite-dimensional space X. Then $\mathcal{L} = L \oplus^{\pi} X$ (see (3.9)) is a Lie algebra and $I = \{0\} \oplus^{\pi} X$ is a Lie ideal of $\mathcal{L}$. It is easy to see that $\mathfrak{P}(\mathcal{L}) = \mathcal{L}$ and $\mathfrak{P}(I) = \{0\}$. Hence, by Theorem 6.19(ii), $R_{\mathrm{Levi}}\,(\mathcal{L}) = \mathcal{L}$ and $R_{\mathrm{Levi}}\,(I) = \{0\}$. Thus $R_{\mathrm{Levi}}\,(I) \neq R_{\mathrm{Levi}}(\mathcal{L}) \cap I = I$.

6.3. **Some extensions of classical radicals.** In this section we consider some Lie ideal-multifunctions Γ on $\mathfrak{L}$ related to commutative and solvable ideals. Although they generate different preradicals S_Γ, the preradicals $S_\Gamma^{\mathbf{s}}$ corresponding to them (see (4.11)) often generate equal radicals that extend the classical radical rad on $\mathbf{L}^{\mathrm{f}}$.

We start with the multifunction $\mathrm{I}^{\mathrm{sol}}$ defined above and the multifunction "Abel":

$$\mathrm{Abel} = \{\mathrm{Abel}_{\mathcal{L}}\}_{\mathcal{L}\in\mathfrak{L}} \text{ where } \mathrm{Abel}_{\mathcal{L}} \text{ is the family of all commutative Lie ideals of } \mathcal{L}.$$

As in Proposition 6.15(i), we have that Abel is a strictly direct and lower stable multifunction (see Definition 5.4), so that S_{Abel} is a lower stable preradical in $\overline{\mathbf{L}}$. Hence $S_{\mathrm{Abel}}^{\mathbf{s}}$ is an under radical (Theorem 4.8(i)) and $(S_{\mathrm{Abel}}^{\mathbf{s}})^*$ is a radical (Corollary 4.9(i)).

Theorem 6.21. (i) $\mathbf{Sem}(S_{\mathrm{I^{sol}}}^{\mathbf{s}}) = \mathbf{Sem}(S_{\mathrm{Abel}}^{\mathbf{s}})$ *and* $\mathcal{L}$ *belongs to them if and only if* $\mathcal{L}$ *has no non-zero commutative finite-dimensional Lie subideals.*

(ii) *The map* $K : \mathcal{L} \mapsto K(\mathcal{L}) = \mathrm{Centre}\,(\mathcal{L})$, *for each* $\mathcal{L} \in \mathfrak{L}$, *is a lower stable preradical and* $(S_{\mathrm{I^{sol}}}^{\mathbf{s}})^* = (S_{\mathrm{Abel}}^{\mathbf{s}})^* = (K^{\mathbf{s}})^*$.

(iii) $(S_{\mathrm{Abel}}^{\mathbf{s}})^*|\mathbf{L}^{\mathrm{f}} = \mathrm{rad}$.

Proof. (i) If J is a Lie subideal of I and I is a Lie subideal of $\mathcal{L}$, then J is a Lie subideal of $\mathcal{L}$.

By (4.11), $\mathcal{L} \in \mathbf{Sem}\,(R^{\mathbf{s}})$ for a preradical R, if and only if $\mathrm{Sub}(\mathcal{L},R) = \{\{0\}\}$, that is,

$$I = \{0\} \text{ is the only Lie subideal of } \mathcal{L} \text{ satisfying } I = R(I). \tag{6.5}$$

Let $\Gamma = \{\Gamma_{\mathcal{L}}\}_{\mathcal{L}\in\mathfrak{L}}$ where each family $\Gamma_{\mathcal{L}} = \{J\colon\ J \lhd \mathcal{L} \text{ and } J \text{ has some property } \mathbf{T}\}$. Let $R = S_\Gamma$, so that $R(\mathcal{L}) \overset{(5.3)}{=} \mathfrak{s}(\Gamma_{\mathcal{L}})$. Then (6.5) is equivalent to the following condition:

$$\mathcal{L} \text{ has no Lie subideals that have property } \mathbf{T}. \tag{6.6}$$

Indeed, let (6.6) do not hold and let $I \neq \{0\}$ be a Lie subideal of $\mathcal{L}$ that has property $\mathbf{T}$. Then $I \in \Gamma_I$, so that $\{0\} \neq I = \mathfrak{s}(\Gamma_I)$. Conversely, let $I \neq \{0\}$ be a Lie subideal of $\mathcal{L}$ such that $I = \mathfrak{s}(\Gamma_I)$. Then $\Gamma_I \neq \{\{0\}\}$, so that I has a Lie ideal $J \neq \{0\}$ that has property $\mathbf{T}$. As J is a Lie subideal of $\mathcal{L}$, (6.6) does not hold. Thus $\mathcal{L} \in \mathbf{Sem}\,(S_{\mathrm{Abel}}^{\mathbf{s}})$ if and only if $\mathcal{L}$ has no non-zero Lie subideals which are finite-dimensional and commutative. Similarly, $\mathcal{L} \in \mathbf{Sem}\left(S_{\mathrm{I^{sol}}}^{\mathbf{s}}\right)$ if and only if $\mathcal{L}$ has no non-zero Lie subideals which are finite-dimensional and solvable.

Let us show that $\mathcal{L}$ has a finite-dimensional solvable Lie subideal $Y \neq \{0\}$ if and only if it has a non-zero finite-dimensional commutative Lie subideal. As $Z = [Y,Y]$ is a nilpotent Lie ideal of Y, it is a Lie subideal of $\mathcal{L}$. If $Z = \{0\}$ then Y is a commutative Lie subideal of $\mathcal{L}$. If $Z \neq \{0\}$ then Z has a non-zero centre which is a commutative Lie subideal of $\mathcal{L}$. The converse statement is obvious. This implies $\mathbf{Sem}(S_{\mathrm{I^{sol}}}^{\mathbf{s}}) = \mathbf{Sem}(S_{\mathrm{Abel}}^{\mathbf{s}})$.

(ii) If $\mathcal{L} \in \mathfrak{L}$ then $[K(\mathcal{L}),\mathcal{L}] = \{0\}$. If $f\colon\ \mathcal{L} \longrightarrow \mathcal{M}$ is a morphism in $\overline{\mathbf{L}}$, then $[f\,(K(\mathcal{L})), f(\mathcal{L})] = f([K(\mathcal{L}),\mathcal{L}]) = \{0\}$, whence $\overline{f\,(K(\mathcal{L}))} \subseteq K(\mathcal{M})$. Thus K is a preradical. As $K(\mathcal{L})$ is commutative, $K\,(K\,(\mathcal{L})) = K\,(\mathcal{L})$, so that K is lower stable.

By (6.5), $\mathcal{L} \in \mathbf{Sem}\,(K^{\mathbf{s}})$ if and only if $I = \{0\}$ is the only Lie subideal of $\mathcal{L}$ satisfying $I = K(I)$. From the definition of K we have that this is possible if and only if $\{0\}$ is the only commutative Lie subideal of $\mathcal{L}$. It means, in turn, that $\{0\}$ is the only finite-dimensional commutative Lie subideal of $\mathcal{L}$. Hence, by (i), $\mathbf{Sem}\left(S_{\mathrm{I^{sol}}}^{\mathbf{s}}\right) = \mathbf{Sem}\,(S_{\mathrm{Abel}}^{\mathbf{s}}) = \mathbf{Sem}\,(K^{\mathbf{s}})$. Applying Theorem 4.5, we have $\mathbf{Sem}\left((S_{\mathrm{I^{sol}}}^{\mathbf{s}})^*\right) = \mathbf{Sem}\,((S_{\mathrm{Abel}}^{\mathbf{s}})^*) = \mathbf{Sem}\,((K^{\mathbf{s}})^*)$. As $(S_{\mathrm{I^{sol}}}^{\mathbf{s}})^*$, $(S_{\mathrm{Abel}}^{\mathbf{s}})^*$ and $(K^{\mathbf{s}})^*$ are radicals, we have from Corollary 3.8 that $(S_{\mathrm{I^{sol}}}^{\mathbf{s}})^* = (S_{\mathrm{Abel}}^{\mathbf{s}})^* = (K^{\mathbf{s}})^*$.

(iii) Let $\mathcal{L} \in \mathfrak{L}^{\mathrm{f}}$. Note that $\mathrm{rad}(\mathcal{L}) = \{0\}$ if and only if $\mathcal{L}$ has no non-zero commutative Lie ideals. Indeed, if $\mathrm{rad}\,(\mathcal{L}) \neq \{0\}$ then $J = [\mathrm{rad}\,(\mathcal{L})\,, \mathrm{rad}\,(\mathcal{L})]$ is a nilpotent Lie ideal of $\mathcal{L}$. If $J = \{0\}$ then $\mathrm{rad}(\mathcal{L})$ is a commutative Lie ideal of $\mathcal{L}$. If $J \neq \{0\}$ then J has a non-zero centre which is a commutative Lie ideal of $\mathcal{L}$. Conversely, let $\mathrm{rad}\,(\mathcal{L}) = \{0\}$. Since $\mathrm{rad}\,(\mathcal{L})$ is the largest solvable Lie ideal, $\mathcal{L}$ has no non-zero commutative Lie ideals.

By the above argument and by Lemma 2.9, $\mathcal{L} \in \mathbf{Sem}\,(\mathrm{rad})$ if and only if $\mathcal{L}$ has no non-zero commutative Lie subideals. Hence, by (i), $\mathbf{Sem}\,(\mathrm{rad}) = \mathbf{Sem}(S_{\mathrm{Abel}}^{\mathbf{s}}|\mathbf{L}^{\mathrm{f}})$. From Theorem 4.5 it follows that $\mathbf{Sem}\,(\mathrm{rad}) = \mathbf{Sem}(S_{\mathrm{Abel}}^{\mathbf{s}}|\mathbf{L}^{\mathrm{f}}) = \mathbf{Sem}((S_{\mathrm{Abel}}^{\mathbf{s}})^*|\mathbf{L}^{\mathrm{f}})$. As $(S_{\mathrm{Abel}}^{\mathbf{s}})^*$ and "rad" are radicals, we have from Corollary 3.8 that $(S_{\mathrm{Abel}}^{\mathbf{s}})^*|\mathbf{L}^{\mathrm{f}} = \mathrm{rad}$. □

F. Vasilescu [V] extended the notion of the classical solvable radical to infinite dimensional Lie algebras $\mathcal{L}$ in the following way. He called a Lie ideal J of $\mathcal{L}$ *primitive* if

$$[A,A] \subseteq J \Longrightarrow A \subseteq J, \tag{6.7}$$

for any Lie ideal A of $\mathcal{L}$. This is equivalent to the condition that $\mathcal{L}/J$ has no abelian Lie ideals. Denote by $R_{\mathcal{L}}$ the intersection of all primitive Lie ideals of $\mathcal{L}$. It was proved in [V] that $R_{\mathcal{L}} = \mathrm{rad}(\mathcal{L})$ if $\mathcal{L}$ is finite-dimensional. Applying this to Banach Lie algebras $\mathcal{L}$ and denoting by $R_{\mathcal{L}}$ the intersection of all *closed* primitive Lie ideals of $\mathcal{L}$, one obtains an upper stable preradical on $\mathbf{L}$. However, it is not clear whether this preradical is balanced and lower stable. The main obstacle is that we don't know whether a Banach Lie algebra, whose Lie ideal has a non-zero commutative Lie ideal, has itself a non-zero commutative Lie ideal.

To avoid this difficulty let us change the definition of the radical as follows. We call a closed Lie subideal (ideal) J of a Banach Lie algebra $\mathcal{L}$ *primitive*, if the implication (6.7) holds for any Lie *subideal* A of $\mathcal{L}$. Set

$$P_V(\mathcal{L}) = \cap\,\{J\colon\ J \text{ is a closed primitive Lie ideal of } \mathcal{L}\}, \tag{6.8}$$
$$I_V(\mathcal{L}) = \cap\,\{J\colon\ J \text{ is a closed primitive Lie subideal of } \mathcal{L}\}.$$

Lemma 6.22. *$P_V(\mathcal{L})$ is a characteristic primitive ideal and $P_V(\mathcal{L}) = I_V(\mathcal{L})$.*

Proof. $P_V(\mathcal{L})$ is a closed Lie ideal of $\mathcal{L}$. If A is a Lie subideal of $\mathcal{L}$ and $[A, A] \subseteq P_V(\mathcal{L})$, then $[A, A] \subseteq J$ for each primitive ideal of $\mathcal{L}$. Hence $A \subseteq J$, so that $A \subseteq P_V(\mathcal{L})$. Thus $P_V(\mathcal{L})$ is primitive. A similar argument shows that $I_V(\mathcal{L})$ is a closed primitive subideal of $\mathcal{L}$.

Clearly, $I_V(\mathcal{L}) \subseteq P_V(\mathcal{L})$. Each bounded isomorphism of $\mathcal{L}$ maps Lie subideals of $\mathcal{L}$ into Lie subideals and primitive Lie subideals into primitive Lie subideals. Hence $I_V(\mathcal{L})$ is invariant for all bounded isomorphisms of $\mathcal{L}$. Therefore, by Lemma 2.2, $I_V(\mathcal{L})$ is a characteristic Lie ideal of $\mathcal{L}$. Thus $I_V(\mathcal{L})$ is a closed primitive Lie ideal of $\mathcal{L}$. By (6.8), $P_V(\mathcal{L}) \subseteq I_V(\mathcal{L})$, so $P_V(\mathcal{L}) = I_V(\mathcal{L})$. $\square$

In the same way as Vasilescu's radical coincides with "rad" in the category $\mathbf{L}^{\mathrm{f}}$ of all finite-dimensional Lie algebras, P_V also coincides with "rad" in $\mathbf{L}^{\mathrm{f}}$.

Lemma 6.23. *$P_V(\mathcal{L}) = \mathrm{rad}(\mathcal{L})$ if $\mathcal{L}$ is finite-dimensional.*

Proof. Note first that $\mathrm{rad}(\mathcal{L})$ is a primitive Lie ideal of $\mathcal{L}$. Indeed, let A be a subideal of $\mathcal{L}$ with $[A, A] \subseteq \mathrm{rad}(\mathcal{L})$. Then $[A, A]$ is a solvable Lie algebra, whence A is a solvable Lie algebra. Let $A \lhd A_1 \lhd A_2 \lhd ... \lhd A_n \lhd \mathcal{L}$. As rad is a radical, $A = \mathrm{rad}(A) \subseteq \mathrm{rad}(A_1) \subseteq ... \subseteq \mathrm{rad}(\mathcal{L})$. Thus $\mathrm{rad}(\mathcal{L})$ is a primitive Lie ideal of $\mathcal{L}$. Hence, by (6.8), $P_V(\mathcal{L}) \subseteq \mathrm{rad}(\mathcal{L})$.

To prove the lemma, it remains to establish the converse inclusion. Let $J = \mathrm{rad}(\mathcal{L})$ and let $J_{[1]} \supseteq J_{[2]} \supseteq ... \supseteq J_{[n]} \subseteq J_{[n+1]} = 0$ be the Lie ideals of $\mathcal{L}$ defined in (6.2). Since $[J_{[n]}, J_{[n]}] = J_{[n+1]} = \{0\} \subseteq P_V(\mathcal{L})$ and $P_V(\mathcal{L})$ is primitive, we have $J_{[n]} \subseteq P_V(\mathcal{L})$. Proceeding in this way, we obtain that $J_{[n-1]} \subseteq P_V(\mathcal{L})$,..., and finally $J \subseteq P_V(\mathcal{L})$. Thus $\mathrm{rad}(\mathcal{L}) = P_V(\mathcal{L})$. $\square$

Our aim is to show that P_V is an over radical on $\mathfrak{L}$ and that the corresponding radical P_V° coincides with $(S^{\mathrm{s}}_{\mathrm{Abel}})^*$.

Proposition 6.24. *P_V is an over radical on $\mathfrak{L}$.*

Proof. Let $f : \mathcal{L} \to \mathcal{M}$ be a continuous homomorphism of Banach Lie algebras and $\overline{f(\mathcal{L})} = \mathcal{M}$. Then $\overline{f(A)}$ is a closed Lie ideal of $\mathcal{M}$, for each Lie ideal A of $\mathcal{L}$, which implies that $\overline{f(A)}$ is a closed Lie subideal of $\mathcal{M}$ if A is a Lie subideal of $\mathcal{L}$.

Let I be a closed primitive Lie ideal of $\mathcal{M}$. Then $J := f^{-1}(I)$ is a closed primitive Lie ideal of $\mathcal{L}$. Indeed, if A is a Lie subideal of $\mathcal{L}$ with $[A, A] \subseteq J$, then $[\overline{f(A)}, \overline{f(A)}] \subseteq I$. Hence $\overline{f(A)} \subseteq I$, so that $A \subseteq J$. By (6.8), $P_V(\mathcal{L}) \subseteq J$, so that $f(P_V(\mathcal{L})) \subseteq I$. Thus $f(P_V(\mathcal{L}))$ is contained in all closed primitive Lie ideals of $\mathcal{M}$, so that $f(P_V(\mathcal{L})) \subseteq P_V(\mathcal{M})$. Therefore P_V is a preradical.

Let I be a closed Lie ideal of $\mathcal{L}$. If A is a Lie subideal of I with $[A, A] \subseteq I \cap P_V(\mathcal{L})$, then $A \subseteq P_V(\mathcal{L})$ because $P_V(\mathcal{L})$ is primitive. Thus $A \subseteq I \cap P_V(\mathcal{L})$, so that $I \cap P_V(\mathcal{L})$ is a primitive ideal of I. This implies that $P_V(I) \subseteq I \cap P_V(\mathcal{L})$. We proved that the preradical P_V is balanced.

Let $q\colon \mathcal{L} \to \mathcal{L}/P_V(\mathcal{L})$ be the quotient map. If K is a commutative Lie subideal of $\mathcal{L}/P_V(\mathcal{L})$, then $F := q^{-1}(K)$ is a Lie subideal of $\mathcal{L}$ and $[F, F] \subseteq P_V(\mathcal{L})$. As $P_V(\mathcal{L})$ is primitive, $F \subseteq P_V(\mathcal{L})$, so that $K = \{0\}$. Thus $\{0\}$ is a primitive Lie ideal in $\mathcal{L}/P_V(\mathcal{L})$ whence $P_V(\mathcal{L}/P_V(\mathcal{L})) = \{0\}$. Thus P_V is upper stable. $\square$

Theorem 6.25. *The radicals* $(S^{\mathrm{s}}_{\mathrm{Abel}})^*$ *and* P°_V *coincide.*

Proof. As P_V is an over radical, P°_V is a radical by Theorem 4.6(ii). Hence, by Corollary 3.8, it suffices to show that $\mathbf{Sem}(P^\circ_V) = \mathbf{Sem}(S^{\mathrm{s}}_{\mathrm{Abel}})^*$. By Theorem 6.21, $\mathcal{L} \in \mathbf{Sem}(S^{\mathrm{s}}_{\mathrm{Abel}})^*$ if and only if $\mathcal{L}$ has no non-zero commutative Lie subideals. If this condition is fulfilled, then $\{0\}$ is a primitive ideal of $\mathcal{L}$, so that $P_V(\mathcal{L}) = \{0\}$ and therefore $P^\circ_V(\mathcal{L}) = \{0\}$. We have to show the converse: if $P^\circ_V(\mathcal{L}) = \{0\}$ then $\mathcal{L}$ has no commutative subideals.

It follows from (6.7) that if A is a commutative Lie subideal of $\mathcal{L}$, then A is contained in each primitive Lie ideal of $\mathcal{L}$, so that $A \subseteq P_V(\mathcal{L})$. Furthermore, A is a commutative Lie ideal of $P_V(\mathcal{L})$. Hence, as above, $A \subseteq P_V(P_V(\mathcal{L}))$. Arguing in this way, one easily shows that $A \subseteq P^\alpha_V(\mathcal{L})$ for each ordinal α, whence $A \subseteq P^\circ_V(\mathcal{L})$. As $P^\circ_V(\mathcal{L}) = \{0\}$, we get $A = \{0\}$. $\square$

Consider now the map D: $\mathcal{L} \mapsto \overline{[\mathcal{L},\mathcal{L}]} = \mathcal{L}_{[1]}$. Then $D^n(\mathcal{L}) = \overline{[\mathcal{L}_{[n]},\mathcal{L}_{[n]}]} = \mathcal{L}_{[n+1]}$ for each $n \in \mathbb{N}$. Defining $D^\alpha(\mathcal{L})$ as in (4.5) for each ordinal α, we obtain the D-superposition series $\{D^\alpha(\mathcal{L})\}$ — a transfinite analogue of the *derived series* of $\mathcal{L}$, and define D° as in (4.6). Set

$$\mathcal{D} = D^\circ. \tag{6.9}$$

Theorem 6.26. (i) *D is an over radical and $\mathcal{D}$ is a radical in* $\overline{\mathbf{L}}$.

(ii) *For each $\mathcal{L} \in \mathfrak{L}$, $\mathcal{D}(\mathcal{L})$ is the largest $\mathcal{D}$-radical Lie ideal of $\mathcal{L}$; in other words it is the largest of all Lie ideals I of $\mathcal{L}$ satisfying $I = \overline{[I,I]}$.*

(iii) *The restriction of $\mathcal{D}$ to $\mathbf{L}^{\mathrm{f}}$ coincides with R_{Levi}.*

Proof. (i) For every $\mathcal{L} \in \mathfrak{L}$, $\overline{f(\mathcal{L}_{[1]})} = \mathcal{M}_{[1]}$ for each morphism f: $\mathcal{L} \longrightarrow \mathcal{M}$ in $\overline{\mathbf{L}}$ and $F(I) = I_{[1]} \subseteq \mathcal{L}_{[1]} = D(\mathcal{L})$ for each $I \lhd \mathcal{L}$. Also $D(\mathcal{L}/D(\mathcal{L})) = \left(\mathcal{L}/\mathcal{L}_{[1]}\right)_{[1]} = 0$. Hence D is a balanced upper stable preradical. Thus D is an over radical. By Theorem 4.6(ii), $D^\circ = \mathcal{D}$ is a radical.

(ii) As $\mathcal{D}$ is a radical, $\mathcal{D}(\mathcal{D}(\mathcal{L})) = \mathcal{D}(\mathcal{L})$. Thus $\mathcal{D}(\mathcal{L})$ is $\mathcal{D}$-radical. On the other hand, the condition $I = \mathcal{D}(I)$ implies $I = \mathcal{D}(I) \subseteq \mathcal{D}(\mathcal{L})$.

(iii) By Theorem 6.19(ii) and (6.4), $R_{\mathrm{Levi}}(\mathcal{L}) = \mathfrak{P}(\mathcal{L}) = \mathcal{D}(\mathcal{L})$ for each $\mathcal{L} \in \mathbf{L}^{\mathrm{f}}$. $\square$

As R_{Levi} is not a hereditary radical, $\mathcal{D}$ is not hereditary too.

Consider now the Lie ideal-multifunction $\mathrm{C} = \{\mathrm{C}_{\mathcal{L}}\}$ on $\mathfrak{L}$ such that each family $\mathrm{C}_{\mathcal{L}}$ consists of closed characteristic Lie ideals $\left\{\mathcal{L}^{[\alpha]}\right\}_\alpha$, where $\mathcal{L}^{[1]} = \mathcal{L}$, $\mathcal{L}^{[\alpha+1]} = \overline{\left[\mathcal{L},\mathcal{L}^{[\alpha]}\right]}$ for each ordinal α, and $\mathcal{L}^{[\alpha]} = \cap_{\alpha'<\alpha}\mathcal{L}^{[\alpha']}$ for limit ordinal α. The series $\left\{\mathcal{L}^{[\alpha]}\right\}_\alpha$ is a transfinite analogue of the *lower central series* of $\mathcal{L}$. As in (5.3), set $P_{\mathrm{C}}(\mathcal{L}) = \mathfrak{p}(\mathrm{C}_{\mathcal{L}}) = \cap_\alpha \mathcal{L}^{[\alpha]}$.

Theorem 6.27. *P_{C} is an over radical and $P^\circ_{\mathrm{C}} = \mathcal{D}$.*

Proof. By induction and by (4.4), $\overline{f\left(\mathcal{L}^{[\alpha]}\right)} \subseteq \mathcal{M}^{[\alpha]}$ for every morphism $f : \mathcal{L} \longrightarrow \mathcal{M}$ in $\overline{\mathbf{L}}$. Hence, by (4.4), P_{C} is a preradical. For each $I \lhd \mathcal{L}$ and α, $I^{[\alpha]} \subseteq \mathcal{L}^{[\alpha]}$. Hence P_{C} is balanced.

Set $I = P_{\mathrm{C}}(\mathcal{L})$. By induction, it is easy to see that $(\mathcal{L}/I)^{[\alpha]} \subseteq \mathcal{L}^{[\alpha]}/I$, whence $P_{\mathrm{C}}(\mathcal{L}/I) = \cap_\alpha(\mathcal{L}/I)^{[\alpha]} \subseteq \cap_\alpha(\mathcal{L}^{[\alpha]}/I) = P_{\mathrm{C}}(\mathcal{L})/I = I/I = \{0\}$. Thus P_{C} is upper stable. Hence P_{C} is an over radical and, by Theorem 4.6, P°_{C} is a radical.

By Corollary 3.8(ii), to prove the equality $P^\circ_{\mathrm{C}} = \mathcal{D}$ it suffices to show that $\mathbf{Rad}(P^\circ_{\mathrm{C}}) = \mathbf{Rad}(\mathcal{D})$. As P_{C} and D are balanced, it follows from Theorem 4.2(i) that we only need to show that $\mathbf{Rad}(P_{\mathrm{C}}) = \mathbf{Rad}(D)$ which is obvious, as $\mathcal{L}$ belongs to any of these classes if and only if $\mathcal{L} = \overline{[\mathcal{L},\mathcal{L}]}$. $\square$

7. Frattini radical

The Frattini theory of finite-dimensional Lie algebras $\mathcal{L}$ studies the structure of maximal Lie ideals and maximal Lie subalgebras in $\mathcal{L}$. To extend it to Banach Lie algebras, one can introduce multifunctions $\mathrm{S}^{\max}_{\mathcal{L}}$ and $\mathrm{J}^{\max}_{\mathcal{L}}$, where $\mathrm{S}^{\max}_{\mathcal{L}}$ (respectively, $\mathrm{J}^{\max}_{\mathcal{L}}$), for each $\mathcal{L} \in \mathfrak{L}$, is the family of *all* maximal proper closed Lie subalgebras (respectively, Lie ideals) of $\mathcal{L}$. It can be shown that $P_{\mathrm{S}^{\max}}$ and $P_{\mathrm{J}^{\max}}$ are upper stable preradicals on $\overline{\mathbf{L}}$. However, this approach encounters serious

obstacles and has not given, so far, any further interesting results. For example, we do not know whether the preradicals $P_{\mathfrak{S}^{\max}}$ and $P_{\mathfrak{J}^{\max}}$ are balanced.

As we will see further, a substantial theory can be developed if, instead of all maximal Lie subalgebras (ideals), one considers maximal Lie subalgebras and ideals *of finite codimension.*

Let us consider the following four Lie subalgebra-multifunctions on $\mathfrak{L}$:

1) $\mathfrak{S} = \{\mathfrak{S}_{\mathcal{L}}\}_{\mathcal{L}\in\mathfrak{L}}$ where $\mathfrak{S}_{\mathcal{L}}$ consists of all *proper* closed Lie subalgebras of finite codimension in $\mathcal{L}$;
2) $\mathfrak{S}^{\max} = \{\mathfrak{S}^{\max}_{\mathcal{L}}\}_{\mathcal{L}\in\mathfrak{L}}$ where $\mathfrak{S}^{\max}_{\mathcal{L}}$ consists of all *maximal proper* closed Lie subalgebras of finite codimension in $\mathcal{L}$;
3) $\mathfrak{J} = \{\mathfrak{J}_{\mathcal{L}}\}_{\mathcal{L}\in\mathfrak{L}}$ where $\mathfrak{J}_{\mathcal{L}}$ consists of all *proper* closed Lie ideals of finite codimension in $\mathcal{L}$;
4) $\mathfrak{J}^{\max} = \{\mathfrak{J}^{\max}_{\mathcal{L}}\}_{\mathcal{L}\in\mathfrak{L}}$ where $\mathfrak{J}^{\max}_{\mathcal{L}}$ consists of all *maximal proper* closed Lie ideals of finite codimension in $\mathcal{L}$.

Proposition 7.1. *Let* Γ *be any of the multifunctions* $\mathfrak{S}$, $\mathfrak{S}^{\max}$, $\mathfrak{J}$, $\mathfrak{J}^{\max}$ *and let* $f:\mathcal{L}\to\mathcal{M}$ *be a morphism in* $\overline{\mathbf{L}}$. *If* $K\in\Gamma_{\mathcal{M}}$ *then* $f^{-1}(K) := f^{-1}(K\cap f(\mathcal{L}))\in\Gamma_{\mathcal{L}}$.

Proof. Since K is a proper closed subspace of finite codimension in $\mathcal{M}$, then $F := f^{-1}(K)$ is a closed proper subspace of finite codimension in $\mathcal{L}$. Clearly, F is a Lie subalgebra, and it is a Lie ideal if K is a Lie ideal.

We claim that F is maximal if K is maximal. Indeed, let $L\subseteq\mathcal{L}$ be a maximal proper closed Lie subalgebra (ideal) containing F. Note that $L = f^{-1}(f(L))$ because $f^{-1}(0)\subseteq F\subseteq L$. By Lemma 6.1, $f(L)+K$ is a closed Lie subalgebra (ideal) in $\mathcal{M}$. If K is maximal then either $f(L)\subseteq K$ or $f(L)+K=\mathcal{M}$. In the first case $L\subseteq f^{-1}(K)=F$. As L is maximal, $F=L$ and F is maximal. In the second case $f(\mathcal{L})\subseteq\mathcal{M}=f(L)+K$, whence $\mathcal{L}\subseteq f^{-1}(f(L))+f^{-1}(K) = L+F=L$, a contradiction. $\square$

Remark 7.2. If f is *onto* then, as above, we have that $L\in\Gamma_{\mathcal{L}}$ implies $f(L)\in\Gamma_{\mathcal{M}}\cup\{\mathcal{M}\}$. Thus

$$\Gamma_{\mathcal{M}}\subseteq\{f(L): L\in\Gamma_{\mathcal{L}}\}\subseteq\{\mathcal{M}\}\cup\Gamma_{\mathcal{M}}. \tag{7.1}$$

Recall (see (4.1), (5.3)) that, for each family Γ of Lie subalgebras of a Banach Lie algebra $\mathcal{L}$,

$$P_\Gamma(\mathcal{L}) = \mathfrak{p}(\Gamma) = \cap_{L\in\Gamma}L, \text{ if } \Gamma\neq\varnothing; \text{ and } P_\Gamma(\mathcal{L})=\mathcal{L}, \text{ if } \Gamma=\varnothing. \tag{7.2}$$

Theorem 7.3. $P_{\mathfrak{S}}$, $P_{\mathfrak{S}^{\max}}$, $P_{\mathfrak{J}}$, $P_{\mathfrak{J}^{\max}}$ *are upper stable preradicals in* $\overline{\mathbf{L}}$.

Proof. Let Γ be any of the multifunctions $\mathfrak{S}$, $\mathfrak{S}^{\max}$, $\mathfrak{J}$, $\mathfrak{J}^{\max}$ and $f\colon\mathcal{L}\to\mathcal{M}$ be a morphism in $\overline{\mathbf{L}}$. If $\Gamma_{\mathcal{M}}=\varnothing$ then $P_\Gamma(\mathcal{M})\overset{(7.2)}{=}\mathcal{M}$, so that $f(P_\Gamma(\mathcal{L}))\subseteq P_\Gamma(\mathcal{M})$.

Let $\Gamma_{\mathcal{M}}\neq\varnothing$. By Proposition 7.1, $\Gamma_{\mathcal{L}}\neq\varnothing$ and if $x\in P_\Gamma(\mathcal{L})=\mathfrak{p}(\Gamma_{\mathcal{L}})=\underset{L\in\Gamma_{\mathcal{L}}}{\cap}L$, then $f(x)\in\underset{M\in\Gamma_{\mathcal{M}}}{\cap}f\left(f^{-1}(M)\right)=\underset{M\in\Gamma_{\mathcal{M}}}{\cap}M=P_\Gamma(\mathcal{M})$. This implies $f(P_\Gamma(\mathcal{L}))\subseteq P_\Gamma(\mathcal{M})$, so that P_Γ is a preradical.

Let $I=P_\Gamma(\mathcal{L})$ and $q\colon\mathcal{L}\to\mathcal{L}/I$ be the quotient map. As $I\subseteq L$, for each $L\in\Gamma_{\mathcal{L}}$, we have

$$\{0\}=q(P_\Gamma(\mathcal{L}))=q\left(\underset{L\in\Gamma_{\mathcal{L}}}{\cap}L\right)=\underset{L\in\Gamma_{\mathcal{L}}}{\cap}q(L)\overset{(7.1)}{=}\underset{M\in\Gamma_{\mathcal{L}/I}}{\cap}M=P_\Gamma(\mathcal{L}/I).$$

Hence P_Γ is upper stable. $\square$

For a subset N of $\mathcal{L}$, denote by $\mathrm{Alg}(N)$ the closed Lie subalgebra and by $\mathrm{Id}(N)$ the closed Lie ideal of $\mathcal{L}$ generated by N. Let F be a family of proper closed Lie ideals of $\mathcal{L}$. We say that $a\in\mathcal{L}$ is an *ideal* F*-nongenerator* if $\mathrm{Id}(L\cup\{a\})\neq\mathcal{L}$ for all $L\in F$. If F is a family of proper closed Lie subalgebras of $\mathcal{L}$, then $a\in\mathcal{L}$ is an F*-nongenerator* if $\mathrm{Alg}(L\cup\{a\})\neq\mathcal{L}$ for all $L\in F$.

Lemma 7.4. *Let* $\mathcal{L}$ *be a Banach Lie algebra. Then* $P_{\mathfrak{S}^{\max}}(\mathcal{L})$ *is the set of all* $\mathfrak{S}_{\mathcal{L}}$*-nongenerators and* $P_{\mathfrak{J}^{\max}}(\mathcal{L})$ *is the set of all ideal* $\mathfrak{J}_{\mathcal{L}}$*-nongenerators.*

Proof. As each $L \in \mathfrak{S}_{\mathcal{L}}$ is contained in some $M \in \mathfrak{S}_{\mathcal{L}}^{\max}$, the sets of $\mathfrak{S}_{\mathcal{L}}$-nongenerators and $\mathfrak{S}_{\mathcal{L}}^{\max}$-nongenerators coincide. If $a \notin P_{\mathfrak{S}^{\max}}(\mathcal{L})$ then $a \notin L$ for some $L \in \mathfrak{S}_{\mathcal{L}}^{\max}$. Hence $\mathrm{Alg}\,(L \cup \{a\}) = \mathcal{L}$ while $L \neq \mathcal{L}$, so that a is not a $\mathfrak{S}_{\mathcal{L}}$-nongenerator. Conversely, if $b \in P_{\mathfrak{S}^{\max}}(\mathcal{L})$ then $\mathrm{Alg}\,(L \cup \{b\}) = L \neq \mathcal{L}$ for all $L \in \mathfrak{S}_{\mathcal{L}}^{\max}$. Hence b is a $\mathfrak{S}_{\mathcal{L}}^{\max}$-nongenerator.

The proof of the result for $P_{\mathfrak{J}^{\max}}(\mathcal{L})$ is identical. $\square$

For $\dim(\mathcal{L}) < \infty$ the above result was proved in [T, Lemma 2.3] (see also [B, 1.7.2]).

Lemma 7.5. *Let $J \lhd \mathcal{L}$ and let I be a maximal proper closed Lie ideal of finite codimension in $\mathcal{L}$. Then either $J \subseteq I$ or $I \cap J$ is a maximal proper closed Lie ideal of finite codimension in J.*

Proof. Let $J \nsubseteq I$. By Lemma 6.1, $(I + J) \lhd \mathcal{L}$. As $I \subsetneqq I + J$, it follows that $I + J = \mathcal{L}$. Set $K = I \cap J$. Then $K \neq J$, K is a closed Lie ideal of $\mathcal{L}$ and, by Lemma 6.1, K has finite codimension in J. If K is not maximal Lie ideal of J, there is a maximal closed proper Lie ideal W of J containing K. Then $[I + W, \mathcal{L}] = [I + W, I + J] \subseteq I + W$, whence $I + W$ is a closed Lie ideal of $\mathcal{L}$. As I is maximal, either $I + W = I$ or $I + W = \mathcal{L}$.

If $I + W = \mathcal{L}$ then $(I + W) \cap J = I \cap J + W = \mathcal{L} \cap J = J$ which is impossible because $K = I \cap J \subsetneqq W$. Hence $I + W = I$, so that $W \subset I$. Thus $W \subset I \cap J = K$, a contradiction. $\square$

Theorem 7.6. *The preradicals $P_{\mathfrak{S}}$, $P_{\mathfrak{S}^{\max}}$, $P_{\mathfrak{J}}$, $P_{\mathfrak{J}^{\max}}$ are balanced, so that they are over radicals.*

Proof. Let $J \lhd \mathcal{L}$. As $P_{\mathfrak{S}^{\max}}$ is a preradical, it follows from Corollary 3.3 that $P_{\mathfrak{S}^{\max}}(J) \lhd \mathcal{L}$. Assume that $P_{\mathfrak{S}^{\max}}(J) \nsubseteq L$ for some $L \in \mathfrak{S}_{\mathcal{L}}^{\max}$. Then $L + P_{\mathfrak{S}^{\max}}(J)$ is a Lie subalgebra of $\mathcal{L}$ larger than L and, by Lemma 6.1, it is closed. As L is a maximal closed Lie subalgebra of $\mathcal{L}$, $L + P_{\mathfrak{S}^{\max}}(J) = \mathcal{L}$. Hence there is a finite-dimensional linear subspace K of $P_{\mathfrak{S}^{\max}}(J)$ such that $\mathcal{L} = L \dotplus K$.

Let $\{e_i\}_{i=1}^m$ be a basis in K. As $K \subseteq J$, we have $J = (L \cap J) \dotplus K$. Then $M = L \cap J$ and all Lie algebras $M_i = \mathrm{Alg}\,(M \cup \{e_1, \ldots, e_i\})$ have finite codimension in J, so that $M, M_i \in \mathfrak{S}_J$. By Lemma 7.4, all e_i are $\mathfrak{S}_J$-nongenerators of J. As $\mathrm{Alg}\,(M_{m-1} \cup \{e_m\}) = J$, we have $M_{m-1} = J$. Repeating this $m-1$ times, we obtain that $M = J$. Hence $J \subseteq L$, so that $P_{\mathfrak{S}^{\max}}(J) \subseteq J \subseteq L$. This contradiction shows that $P_{\mathfrak{S}^{\max}}(J) \subseteq L$, for all $L \in \mathfrak{S}_{\mathcal{L}}^{\max}$, so that $P_{\mathfrak{S}^{\max}}(J) \subseteq P_{\mathfrak{S}^{\max}}(\mathcal{L})$. Thus $P_{\mathfrak{S}^{\max}}$ is balanced.

By Lemma 7.5, $\mathfrak{J}_J^{\max} \cup \{J\} \overleftarrow{\subset} \{I \cap J : I \in \mathfrak{J}_{\mathcal{L}}^{\max}\} \overleftarrow{\subset} \mathfrak{J}_{\mathcal{L}}^{\max}$ (see (5.1)).

For $\Gamma = \mathfrak{S}$ or $\mathfrak{J}$, the condition $\Gamma_J \cup \{J\} \overleftarrow{\subset} \{I \cap J : I \in \Gamma_{\mathcal{L}}\} \overleftarrow{\subset} \Gamma_{\mathcal{L}}$ follows from Lemma 6.1.

This implies (see (5.2)) that $P_\Gamma(J) = \mathfrak{p}(\Gamma_J) \subseteq \mathfrak{p}(\Gamma_{\mathcal{L}}) = P_\Gamma(\mathcal{L})$, for $\Gamma = \mathfrak{J}^{\max}$, $\mathfrak{S}$ and $\mathfrak{J}$. $\square$

Recall that $\mathcal{L}^{[2]} = \mathcal{L}_{[1]} = \overline{[\mathcal{L}, \mathcal{L}]}$. The following proposition extends to the infinite-dimensional case some results due to Marshall [M, Lemma, p. 420, and Theorem, p. 422.].

Proposition 7.7. *Let $\mathcal{L}$ be a Banach Lie algebra and $Z_{\mathcal{L}}$ be its centre. Then*

(i) *$P_{\mathfrak{J}^{\max}}(\mathcal{L}) \subseteq \mathcal{L}_{[1]}$ and $\mathcal{L}_{[1]} \cap Z_{\mathcal{L}} \subseteq P_{\mathfrak{S}^{\max}}(\mathcal{L})$.*
(ii) *If $\mathcal{L}$ is solvable then $\mathcal{L}_{[1]} = P_{\mathfrak{J}^{\max}}(\mathcal{L})$.*
(iii) *If $\mathcal{L} = I \oplus J$ then $P_{\mathfrak{S}^{\max}}(\mathcal{L}) = P_{\mathfrak{S}^{\max}}(I) \oplus P_{\mathfrak{S}^{\max}}(J)$.*

Proof. (i) Let $I = \mathcal{L}_{[1]}$. Then $\mathcal{L}/I$ is a commutative Banach Lie algebra. As each closed subspace of codimension one is a maximal closed Lie ideal of $\mathcal{L}/I$, we have $P_{\mathfrak{J}^{\max}}(\mathcal{L}/I) = \{0\}$. Hence, by Lemma 3.6(ii), $P_{\mathfrak{J}^{\max}}^{\circ}(\mathcal{L}) \subseteq I = \mathcal{L}_{[1]}$.

Set $J = P_{\mathfrak{S}^{\max}}(\mathcal{L})$. Let $\mathcal{M} = \mathcal{L}/J$, $Z_{\mathcal{M}}$ be its centre and $q\colon \mathcal{L} \to \mathcal{M}$ the quotient map. Then

$$q\left(\mathcal{L}_{[1]} \cap Z_{\mathcal{L}}\right) \subseteq q(\mathcal{L}_{[1]}) \cap q\left(Z_{\mathcal{L}}\right) \subseteq \mathcal{M}_{[1]} \cap Z_{\mathcal{M}}$$

and it suffices to show that $\mathcal{M}_{[1]} \cap Z_{\mathcal{M}} = \{0\}$. Assume, to the contrary, that $\mathcal{M}_{[1]} \cap Z_{\mathcal{M}}$ contains $a \neq 0$. By Theorem 7.3, $P_{\mathfrak{S}^{\max}}$ is upper stable. Hence $P_{\mathfrak{S}^{\max}}(\mathcal{M}) = \{0\}$, so that there is $M \in \mathfrak{S}_{\mathcal{M}}^{\max}$ such that $a \notin M$. As $a \in Z_{\mathcal{M}}$, we have $[a, \mathcal{M}] = 0$, so that $M + \mathbb{C}a$ is a closed Lie subalgebra of finite codimension in $\mathcal{M}$ larger than M. As M is maximal, $M + \mathbb{C}a = \mathcal{M}$ and $\mathcal{M}_{[1]} = \overline{[M + \mathbb{C}a, M + \mathbb{C}a]} = \overline{[M, M]} \subseteq M$. Hence $a \in M$, a contradiction. Thus $\mathcal{M}_{[1]} \cap Z_{\mathcal{M}} = \{0\}$.

(ii) Let $\mathcal{L}$ be solvable. For each $J \in \mathfrak{J}_{\mathcal{L}}^{\max}$, the finite-dimensional Lie algebra $\mathcal{L}/J$ is solvable and has no non-zero Lie ideals. Hence $\dim(\mathcal{L}/J) = 1$, so that $[\mathcal{L},\mathcal{L}] \subseteq J$. Thus $\mathcal{L}_{[1]} = \overline{[\mathcal{L},\mathcal{L}]} \subseteq \bigcap_{J\in\mathfrak{J}_{\mathcal{L}}^{\max}} J = P_{\mathfrak{J}^{\max}}(\mathcal{L})$. Using (i), we have $\mathcal{L}_{[1]} = P_{\mathfrak{J}^{\max}}(\mathcal{L})$.

(iii) follows from Theorem 7.6 and Proposition 3.11(ii) 1). $\square$

By Theorem 4.6, $P_{\mathfrak{S}}^{\circ}$, $P_{\mathfrak{S}^{\max}}^{\circ}$, $P_{\mathfrak{J}}^{\circ}$, $P_{\mathfrak{J}^{\max}}^{\circ}$ are radicals. We will see now that they all coincide.

Theorem 7.8. (i) *For every Banach Lie algebra* $\mathcal{L}$,

$$P_{\mathfrak{S}}(\mathcal{L}) \subseteq P_{\mathfrak{J}}(\mathcal{L}) \subseteq P_{\mathfrak{S}^{\max}}(\mathcal{L}) \subseteq P_{\mathfrak{J}^{\max}}(\mathcal{L}), \tag{7.3}$$

so that $P_{\mathfrak{S}} \le P_{\mathfrak{J}} \le P_{\mathfrak{S}^{\max}} \le P_{\mathfrak{J}^{\max}}$. *If* $\mathfrak{S}_{\mathcal{L}} \neq \varnothing$ *then* $P_{\mathfrak{J}^{\max}}(\mathcal{L}) \neq \mathcal{L}$.

(ii) *The radicals* $P_{\mathfrak{S}}^{\circ}$, $P_{\mathfrak{J}}^{\circ}$, $P_{\mathfrak{S}^{\max}}^{\circ}$ *and* $P_{\mathfrak{J}^{\max}}^{\circ}$ *coincide.*

(iii) $r_{P_{\mathfrak{S}}}^{\circ}(\mathcal{L}) \le r_{P_{\mathfrak{J}}}^{\circ}(\mathcal{L}) \le r_{P_{\mathfrak{S}^{\max}}}^{\circ}(\mathcal{L}) \le r_{P_{\mathfrak{J}^{\max}}}^{\circ}(\mathcal{L})$ (*see* (4.6)), *for each* $\mathcal{L} \in \mathfrak{L}$.

Proof. We begin with the proof of the last statement in part (i). Suppose that $\mathfrak{S}_{\mathcal{L}} \neq \varnothing$. It follows from Theorem 2.5 that $\mathfrak{J}_{\mathcal{L}} \neq \varnothing$, whence $\mathfrak{J}_{\mathcal{L}}^{\max} \neq \varnothing$. Hence $P_{\mathfrak{J}^{\max}}(\mathcal{L}) \neq \mathcal{L}$.

If $\mathfrak{S}_{\mathcal{L}}^{\max} = \varnothing$, then $\mathfrak{J}_{\mathcal{L}}^{\max} = \mathfrak{J}_{\mathcal{L}} = \mathfrak{S}_{\mathcal{L}} = \varnothing$, and it follows from (4.1) and (5.3) that $P_{\Gamma}(\mathcal{L}) = \mathfrak{p}(\Gamma_{\mathcal{L}}) = \mathcal{L}$, where Γ is any of these multifunctions. Hence in this case (7.3) holds.

Let $\mathfrak{S}_{\mathcal{L}}^{\max} \neq \varnothing$. Then $\mathfrak{S}_{\mathcal{L}} \neq \varnothing$ and, by the above, $P_{\mathfrak{J}^{\max}}(\mathcal{L}) \neq \mathcal{L}$. As $\mathfrak{J}_{\mathcal{L}} \subseteq \mathfrak{S}_{\mathcal{L}}$, we have $P_{\mathfrak{S}}(\mathcal{L}) = \mathfrak{p}(\mathfrak{S}_{\mathcal{L}}) \subseteq \mathfrak{p}(\mathfrak{J}_{\mathcal{L}}) = P_{\mathfrak{J}}(\mathcal{L})$.

By Theorem 2.5(i), each $M \in \mathfrak{S}_{\mathcal{L}}^{\max}$ contains a closed Lie ideal J of $\mathcal{L}$ of finite codimension. This means that $\mathfrak{J}_{\mathcal{L}} \overleftarrow{\subset} \mathfrak{S}_{\mathcal{L}}^{\max}$. Therefore, by (5.2), $P_{\mathfrak{J}}(\mathcal{L}) \subseteq P_{\mathfrak{S}^{\max}}(\mathcal{L})$.

To prove the last inclusion in (7.3), set $I = P_{\mathfrak{S}^{\max}}(\mathcal{L})$. For each $J \in \mathfrak{J}_{\mathcal{L}}^{\max}$, there is $M \in \mathfrak{S}_{\mathcal{L}}^{\max}$ such that $J \subseteq M$. Then $I \subseteq M$, so that $I + J \subseteq M$. Thus $I + J$ is a proper Lie ideal of $\mathcal{L}$ and, by Lemma 6.1, it is closed. As $J \subseteq I + J$ and J is a maximal closed Lie ideal of $\mathcal{L}$, we have $J = I + J$. Hence $I \subseteq J$. Therefore $P_{\mathfrak{S}^{\max}}(\mathcal{L}) = I \subseteq \bigcap_{J\in\mathfrak{J}_{\mathcal{L}}^{\max}} J = P_{\mathfrak{J}^{\max}}(\mathcal{L})$. Part (i) is proved.

As $P_{\mathfrak{S}}(\mathcal{L}), P_{\mathfrak{J}}(\mathcal{L}), P_{\mathfrak{S}^{\max}}(\mathcal{L}), P_{\mathfrak{J}^{\max}}(\mathcal{L})$ are Lie ideals of $\mathcal{L}$ and the preradicals are balanced, it follows by induction from (7.3) that

$$P_{\mathfrak{S}}^{\alpha}(\mathcal{L}) \subseteq P_{\mathfrak{J}}^{\alpha}(\mathcal{L}) \subseteq P_{\mathfrak{S}^{\max}}^{\alpha}(\mathcal{L}) \subseteq P_{\mathfrak{J}^{\max}}^{\alpha}(\mathcal{L}), \text{ for all } \mathcal{L} \in \mathfrak{L} \tag{7.4}$$

and all ordinal α. This implies

$$P_{\mathfrak{S}}^{\circ}(\mathcal{L}) \subseteq P_{\mathfrak{J}}^{\circ}(\mathcal{L}) \subseteq P_{\mathfrak{S}^{\max}}^{\circ}(\mathcal{L}) \subseteq P_{\mathfrak{J}^{\max}}^{\circ}(\mathcal{L}). \tag{7.5}$$

Set $J = P_{\mathfrak{S}}^{\circ}(\mathcal{L})$. As $P_{\mathfrak{S}}^{\circ}$ is a radical, it is upper stable. Hence $\{0\} = P_{\mathfrak{S}}^{\circ}(\mathcal{L}/J) \subseteq P_{\mathfrak{J}^{\max}}^{\circ}(\mathcal{L}/J)$.

Set $I = P_{\mathfrak{J}^{\max}}^{\circ}(\mathcal{L}/J)$ and assume that $I \neq \{0\}$. As $P_{\mathfrak{S}}^{\circ}$ is balanced and as $I \lhd \mathcal{L}/J$ and $P_{\mathfrak{S}}^{\circ}(\mathcal{L}/J) = \{0\}$, we have $P_{\mathfrak{S}}^{\circ}(I) = \{0\}$. Hence, by (4.5), $P_{\mathfrak{S}}(I) \neq I$, whence the family $\mathfrak{S}_I \neq \varnothing$. Hence, by (i), $P_{\mathfrak{J}^{\max}}(I) \neq I$. Therefore $P_{\mathfrak{J}^{\max}}^{\circ}(I) \subseteq P_{\mathfrak{J}^{\max}}(I) \neq I$. On the other hand, as $P_{\mathfrak{J}^{\max}}^{\circ}$ is a radical, $P_{\mathfrak{J}^{\max}}^{\circ}(I) = I$. Thus $I = \{0\}$. Therefore it follows from Lemma 3.6(ii) that $P_{\mathfrak{J}^{\max}}^{\circ}(\mathcal{L}) \subseteq J = P_{\mathfrak{S}}^{\circ}(\mathcal{L})$. Taking into account (7.5), we complete the proof of (ii).

(iii) Denote temporarily by P the common radical constructed in (ii). Let $\beta = r_{P_{\mathfrak{J}^{\max}}}^{\circ}(\mathcal{L})$. It follows from (4.6) and (7.4) that $P(\mathcal{L}) \subseteq P_{\mathfrak{S}^{\max}}^{\beta}(\mathcal{L}) \subseteq P_{\mathfrak{J}^{\max}}^{\beta}(\mathcal{L}) = P(\mathcal{L})$. Hence $P(\mathcal{L}) = P_{\mathfrak{S}^{\max}}^{\beta}(\mathcal{L})$, so that $r_{P_{\mathfrak{S}^{\max}}}^{\circ}(\mathcal{L}) \le \beta = r_{P_{\mathfrak{J}^{\max}}}^{\circ}(\mathcal{L})$. The proof of other inequalities is identical. $\square$

Definition 7.9. *We denote by* $\mathcal{F}$ *the common radical in Theorem*7.8(ii):

$$\mathcal{F} = P_{\mathfrak{S}}^{\circ} = P_{\mathfrak{J}}^{\circ} = P_{\mathfrak{S}^{\max}}^{\circ} = P_{\mathfrak{J}^{\max}}^{\circ},$$

and call it the Frattini radical. $\mathcal{F}$*-radical Banach Lie algebras are called also* Frattini-radical.

As $P_{\mathfrak{S}}$, $P_{\mathfrak{S}^{\max}}$, $P_{\mathfrak{J}}$, $P_{\mathfrak{J}^{\max}}$ are balanced preradicals (Theorem 7.6), it follows from Theorem 4.2(ii) that

$$\mathbf{Rad}(\mathcal{F}) = \mathbf{Rad}(P_{\mathfrak{S}}) = \mathbf{Rad}(P_{\mathfrak{S}^{\max}}) = \mathbf{Rad}(P_{\mathfrak{J}}) = \mathbf{Rad}(P_{\mathfrak{J}^{\max}}). \tag{7.6}$$

Thus (see (7.2)) a Banach Lie algebra is $\mathcal{F}$-radical if and only if it satisfies one of the following equivalent conditions:

a) it has no proper closed subalgebras of finite codimension;

b) it has no proper closed ideals of finite codimension;

c) it has no maximal proper closed subalgebras of finite codimension;

d) it has no maximal proper closed ideals of finite codimension.

Corollary 7.10. *A Banach Lie algebra $\mathcal{L}$ has closed Lie subalgebras of finite codimension if and only if it has closed Lie ideals of finite codimension.*

Proof. The condition that $\mathcal{L}$ has no closed Lie ideals of finite codimension means $\mathfrak{J}_{\mathcal{L}} = \varnothing$. Then

$$\mathfrak{J}_{\mathcal{L}} = \varnothing \overset{(7.2)}{\Longleftrightarrow} P_{\mathfrak{J}}(\mathcal{L}) = \mathcal{L} \Longleftrightarrow \mathcal{L} \in \mathbf{Rad}(P_{\mathfrak{J}}) \overset{(7.6)}{=} \mathcal{L} \in \mathbf{Rad}(P_{\mathfrak{S}}) \Longleftrightarrow P_{\mathfrak{S}}(\mathcal{L}) = \mathcal{L} \overset{(7.2)}{\Longleftrightarrow} \mathfrak{S}_{\mathcal{L}} = \varnothing.$$

Hence $\mathcal{L}$ has no closed Lie subalgebras of finite codimension. $\square$

Murphy and Radjavi [MR] proved that the algebra $C(H)$ of all compact operators on a separable Hilbert space H and Schatten ideals C_p of $B(H)$, for $p \geq 2$, have no proper closed Lie subalgebras of finite codimension. In [BKS] Brešar, Kissin and Shulman established that simple Banach associative algebras with trivial centre and without tracial functionals (in particular, $C(H)$ and *all* Schatten ideals C_p, $1 < p < \infty$) have no proper closed Lie ideals. From Corollary 7.10 it follows that they also have no proper closed Lie subalgebras of finite codimension.

The following result can be considered as an "external" application of the radical technique.

Corollary 7.11. (i) *Each Banach Lie algebra $\mathcal{L}$ has the largest closed Lie ideal $\mathcal{F}(\mathcal{L})$ that satisfies one and, therefore, all the above conditions* a)–d); *this ideal is characteristic.*

(ii) *Let $I \lhd \mathcal{L}$. If I and $\mathcal{L}/I$ satisfy conditions* a)–d) *then the same is true for $\mathcal{L}$.*

Proof. As $\mathcal{F}$ is a radical, $\mathcal{F}(\mathcal{L})$ is a characteristic Lie ideal. As $\mathcal{F}(\mathcal{L}) \in \mathbf{Rad}(\mathcal{F})$, it satisfies conditions a)-d). The rest of (i) follows from Corollary 3.8(i). Part (ii) from Lemma 3.6(iv). $\square$

In the next theorem we compare the Frattini radical $\mathcal{F}$ and the radical $\mathcal{D}$ (see (6.9)).

Theorem 7.12. $\mathbf{Sem}\,(\mathcal{D}) \subsetneqq \mathbf{Sem}(\mathcal{F})$, $\mathbf{Rad}\,(\mathcal{F}) \subsetneqq \mathbf{Rad}(\mathcal{D})$ *and* $\mathcal{F} < \mathcal{D}$.

Proof. Recall (see Theorem 6.26) that $D(\mathcal{L}) = \mathcal{L}_{[1]}$ and $\mathcal{D}\,(\mathcal{L}) = D^{\circ}(\mathcal{L}) = \cap\mathcal{L}_{[\alpha]}$ for $\mathcal{L} \in \mathfrak{L}$. By Proposition 7.7(i), $P_{\mathfrak{J}^{\max}}\,(\mathcal{L}) \subseteq \mathcal{L}_{[1]} = D\,(\mathcal{L})$. This implies $P^{\alpha}_{\mathfrak{J}^{\max}}\,(\mathcal{L}) \subseteq \mathcal{L}_{[\alpha]} = D^{\alpha}(\mathcal{L})$ for all α. Hence, by Theorem 7.8(ii), $\mathcal{F} \leq \mathcal{D}$, so that $\mathbf{Sem}\,(\mathcal{D}) \subseteq \mathbf{Sem}(\mathcal{F})$.

Each finite-dimensional semisimple Lie algebra $\mathcal{L}$ belongs to $\mathbf{Sem}(\mathcal{F})$, as $\{0\}$ is a Lie ideal of finite codimension. However, $D^{\circ}(\mathcal{L}) = \mathcal{L}$, as $D\,(\mathcal{L}) = [\mathcal{L},\mathcal{L}] = \mathcal{L}$. Hence $\mathcal{L} \notin \mathbf{Sem}\,(\mathcal{D})$. Thus $\mathbf{Sem}\,(\mathcal{D}) \neq \mathbf{Sem}(\mathcal{F})$. By Proposition 3.7 and Corollary 3.8, $\mathbf{Rad}\,(\mathcal{F}) \subsetneqq \mathbf{Rad}(\mathcal{D})$ and $\mathcal{F} < \mathcal{D}$. $\square$

Corollary 7.13. *Let $\mathcal{L}$ be a Banach Lie algebra. Then $\mathcal{L}/\mathcal{D}\,(\mathcal{L})$ is $\mathcal{F}$-semisimple.*

Proof. The Lie algebra $\mathcal{L}/\mathcal{D}\,(\mathcal{L})$ is $\mathcal{D}$-semisimple and hence $\mathcal{F}$-semisimple by Theorem 7.12. $\square$

We will consider now some examples of $\mathcal{F}$-semisimple algebras.

Example 7.14. (i) *Each finite-dimensional Lie algebra is $\mathcal{F}$-semisimple.* This follows from the fact that $\{0\}$ is a Lie ideal of finite codimension.

(ii) *Each solvable Banach Lie algebra $\mathcal{L}$ is $\mathcal{F}$-semisimple.* This follows from Theorem 7.12. If $\mathcal{L}$ is commutative then, in addition, we have from (7.3) and Proposition 7.7(i) that

$$\mathcal{F}(\mathcal{L}) = P_{\mathfrak{S}}\,(\mathcal{L}) = P_{\mathfrak{J}}\,(\mathcal{L}) = P_{\mathfrak{S}^{\max}}\,(\mathcal{L}) = P_{\mathfrak{J}^{\max}}\,(\mathcal{L}) = \{0\}. \tag{7.7}$$

(iii) *Let X be a Banach space, L be a Lie subalgebra of $B(X)$ and let $\mathcal{L} = L \oplus^{\mathrm{id}} X$ (see* (3.10)). *If L is $\mathcal{F}$-semisimple then $\mathcal{L}$ is $\mathcal{F}$-semisimple.* Indeed, by Proposition 3.11(ii) 3) and the above example, $\mathcal{F}(\mathcal{L}) = \{0\} \oplus^{\mathrm{id}} \mathcal{F}(X) = \{0\}$, so that $\mathcal{L}$ is $\mathcal{F}$-semisimple.

Each infinite-dimensional, topologically simple Banach Lie algebra $\mathcal{L}$ is $\mathcal{F}$-radical, since $\mathfrak{J}_{\mathcal{L}} = \varnothing$, so that (see (7.2) and (7.6)) $\mathcal{L} \in \mathbf{Rad}(P_{\mathfrak{J}}) = \mathbf{Rad}(\mathcal{F})$. For example, the Lie algebra of all nuclear operators with zero trace on a Hilbert space H with respect to the usual Lie product $[a, b] = ab - ba$ is topologically simple (see [BKS, Theorem 5.8]) and therefore $\mathcal{F}$-radical.

Examples of $\mathcal{F}$-radical Banach Lie algebras can be found also among Lie algebras which are far from being simple.

Example 7.15. *The Banach Lie algebra $K_{\mathfrak{N}}$ of all compact operators preserving a given continuous nest $\mathfrak{N}$ of subspaces in H is $\mathcal{F}$-radical.*

To see that $K_{\mathfrak{N}}$ is $\mathcal{F}$-radical, we will show that it has no closed Lie ideals of finite codimension. Assume, to the contrary, that J is such a Lie ideal. All operators in $K_{\mathfrak{N}}$ are quasinilpotent (see [Ri]), so that all operators $\mathrm{ad}(a)$ are quasinilpotent on $K_{\mathfrak{N}}$ and induce quasinilpotent operators on its quotients. Then $L = K_{\mathfrak{N}}/J$ is a nilpotent finite-dimensional Lie algebra. Therefore $[L, L] \neq L$, whence $\overline{[K_{\mathfrak{N}}, K_{\mathfrak{N}}]} \neq K_{\mathfrak{N}}$. On the other hand, each rank one operator $a = e \otimes f$ in $K_{\mathfrak{N}}$ belongs to $\overline{[K_{\mathfrak{N}}, K_{\mathfrak{N}}]}$. Indeed, by [Da, Lemma 3.7], there is a projection p on a subspace in $\mathfrak{N}$ with $pe = e$ and $pf = 0$. Hence $a = pa - ap$. Since projections on subspaces in $\mathfrak{N}$ belong to the strong closure of $K_{\mathfrak{N}}$ ([Da, Lemma 3.9]), a is the norm limit of a sequence $b_n a - ab_n \in [K_{\mathfrak{N}}, K_{\mathfrak{N}}]$. Thus $\overline{[K_{\mathfrak{N}}, K_{\mathfrak{N}}]}$ contains all rank one operators. It remains to note that rank one operators generate $K_{\mathfrak{N}}$ by [Da, Corollary 3.12 and Proposition 3.8], whence $K_{\mathfrak{N}} = \overline{[K_{\mathfrak{N}}, K_{\mathfrak{N}}]}$, a contradiction. $\square$

Example 7.16. *The Frattini radical is not hereditary* (*see* (3.6)).

The Calkin algebra $\mathcal{C} = \mathcal{B}(H)/\mathcal{K}(H)$ is a simple C^*-algebra. Considered as a Lie algebra with respect to the Lie product $[a, b] = ab - ba$, it has only one non-zero Lie ideal $J = \mathbb{C}e$, where e is the unit of $\mathcal{C}$ (since $[\mathcal{C}, \mathcal{C}] = \mathcal{C}$, this follows from Herstein's [H] description of Lie ideals in simple associative algebras). Hence $\mathcal{C}$ is $\mathcal{F}$-radical: $\mathcal{F}(\mathcal{C}) = \mathcal{C}$. On the other hand, J is $\mathcal{F}$-semisimple, since it is finite-dimensional. Hence $\{0\} = \mathcal{F}(J) \neq J \cap \mathcal{F}(\mathcal{C}) = J \cap \mathcal{C} = J$. $\square$

Proposition 7.17. *Let A be a simple infinite-dimensional Banach associative algebra. If its centre $Z_A = \{0\}$ then $\mathcal{F}(A) = \overline{[A, A]}$.*

Proof. As A is a simple Banach algebra, each closed Lie ideal J of A either contains the Lie ideal $C_A := \overline{[A, A]}$, or is contained in Z_A (see [H] and [BKS, Theorem 2.5]). As $Z_A = \{0\}$, either $C_A \subseteq J$ or $J = \{0\}$. Hence, since $\dim A = \infty$, we have that each $J \in \mathfrak{J}_A$ contains C_A. Thus $C_A \subseteq \cap_{J \in \mathfrak{J}_A} J = P_{\mathfrak{J}}(A)$. On the other hand, each subspace of finite codimension containing C_A is a closed Lie ideal of A. As the intersection of such subspaces is C_A, we have $P_{\mathfrak{J}}(A) \subseteq C_A$. Hence $P_{\mathfrak{J}}(A) = C_A$. It remains to prove that $\mathcal{F}(A) = P_{\mathfrak{J}}(A)$. For this we only have to show that C_A has no closed Lie ideals of finite codimension.

It is well known (see, for example, the proof of [BKS, Proposition 2.4]) that $[a, [a, x]] = 0$, for all $x \in A$, implies $a \in Z_A$. In our case this can be written in the form

$$[a, [a, x]] = 0, \text{ for all } x \in A, \text{ implies } a = 0. \tag{7.8}$$

Note that (7.8) implies that A has no commutative Lie ideals. Indeed, if $a \in I$, where I is a commutative Lie ideal, then $[a, [a, x]] = 0$ for all $x \in A$, so that $a = 0$.

It follows also from (7.8) that $\dim C_A = \infty$. Indeed, otherwise, as $\dim A = \infty$, the map $x \in A \to \mathrm{ad}(x)|_{C_A}$ has a non-trivial kernel. Thus there is a non-zero $a \in A$ such that $[a, [y, z]] = 0$, for all $y, z \in A$. This contradicts (7.8).

Thus we have that $C_A = P_{\mathfrak{J}}(A)$ is a non-commutative characteristic Lie ideal of A and $\dim C_A = \infty$. If C_A has a proper closed Lie ideal of finite codimension, it follows from Theorem 2.6 that C_A has a proper closed characteristic ideal J of finite codimension, so that $J \lhd^{\mathrm{ch}} C_A \lhd^{\mathrm{ch}} A$. Hence, by Lemma 2.4, J is a Lie ideal of A. As $J \subsetneqq C_A$, we have $J \subseteq Z_A = \{0\}$. As $\dim C_A = \infty$, $\{0\}$ is not a Lie ideal of finite codimension in C_A, a contradiction. $\square$

It follows from (7.6) that, for the preradicals $P_{\mathfrak{S}}, P_{\mathfrak{J}}, P_{\mathfrak{S}^{\max}}, P_{\mathfrak{J}^{\max}}$ and the radical $\mathcal{F}$, the classes of their radical Lie algebras coincide. We will finish this section by showing that the classes of their semisimple Lie algebras differ.

Recall that $\mathbf{Sem}(\mathcal{F}) = \{\mathcal{L} \in \mathfrak{L}\colon \mathcal{F}(\mathcal{L}) = \{0\}\}$ is the class of $\mathcal{F}$-semisimple Banach Lie algebras. Consider also the classes of semisimple algebras for the preradicals $P_{\mathfrak{S}}, P_{\mathfrak{J}}, P_{\mathfrak{S}^{\max}}, P_{\mathfrak{J}^{\max}}$:

$$\mathbf{Sem}(P_{\mathfrak{S}}) = \{\mathcal{L} \in \mathfrak{L}\colon \cap_{L\in\mathfrak{S}_{\mathcal{L}}} L = \{0\}\}, \qquad \mathbf{Sem}(P_{\mathfrak{J}}) = \{\mathcal{L} \in \mathfrak{L}\colon \cap_{L\in\mathfrak{J}_{\mathcal{L}}} L = \{0\}\},$$
$$\mathbf{Sem}(P_{\mathfrak{S}^{\max}}) = \{\mathcal{L} \in \mathfrak{L}\colon \cap_{L\in\mathfrak{S}^{\max}_{\mathcal{L}}} L = \{0\}\}, \qquad \mathbf{Sem}(P_{\mathfrak{J}^{\max}}) = \{\mathcal{L} \in \mathfrak{L}\colon \cap_{L\in\mathfrak{J}^{\max}_{\mathcal{L}}} L = \{0\}\}.$$

By (7.3), $\mathbf{Sem}(P_{\mathfrak{J}^{\max}}) \subseteq \mathbf{Sem}(P_{\mathfrak{S}^{\max}}) \subseteq \mathbf{Sem}(P_{\mathfrak{J}}) \subseteq \mathbf{Sem}(P_{\mathfrak{S}}) \subseteq \mathbf{Sem}(\mathcal{F})$. We will show that

$$\mathbf{Sem}(P_{\mathfrak{J}^{\max}}) \subsetneqq \mathbf{Sem}(P_{\mathfrak{S}^{\max}}) \subsetneqq \mathbf{Sem}(P_{\mathfrak{J}}) \subsetneqq \mathbf{Sem}(P_{\mathfrak{S}}) \subsetneqq \mathbf{Sem}(\mathcal{F}). \tag{7.9}$$

Firstly, in the following theorem we establish that $\mathbf{Sem}(P_{\mathfrak{S}}) \neq \mathbf{Sem}(\mathcal{F})$.

Let T be a bounded operator on a Banach space X (an example of such operator can be found in [HL]) whose lattice of invariant subspaces $\mathrm{Lat}(T)$ has the following properties:

C_1) the subspaces of finite codimension in $\mathrm{Lat}(T)$ are linearly ordered by inclusion,
C_2) their intersection $X_\omega \neq \{0\}$.

Lemma 7.18. *Let $T \in \mathcal{B}(X)$ and $\mathrm{Lat}(T)$ satisfy C_1) and C_2). If $p \neq 0$ is a polynomial then*

(i) *the closure of the range of $p(T)$ has finite codimension*;
(ii) *each closed subspace Y of finite codimension in X which is invariant for $p(T)$, contains a closed subspace of finite codimension which is invariant for T.*

Proof. (i) Suppose that $\operatorname{codim}(\overline{p(T)X}) = \infty$. As $p(T) = (T - \lambda_1)\cdots(T - \lambda_n)$, it follows that $\operatorname{codim}(\overline{(T-\lambda_k)X}) = \infty$ for some k. All subspaces that contain $\overline{(T-\lambda_k)X}$ are invariant for $T - \lambda_k$ and hence for T. This contradicts the assumption that finite-codimensional subspaces in $\mathrm{Lat}(T)$ are linearly ordered.

(ii) The operator $p(T)$ induces an algebraic operator on X/Y because $\dim(X/Y) < \infty$. Thus there is a polynomial $q(t)$ such that $q(p(T))X \subset Y$. By (i), $\operatorname{codim}(\overline{q(p(T))X}) < \infty$. Since $q(p(T))X$ is invariant for T, we are done. $\square$

Theorem 7.19. *Let $T \in \mathcal{B}(X)$ with $\mathrm{Lat}(T)$ satisfying C_1) and C_2). Let A be the Banach algebra of operators generated by T, and let $\mathcal{L} = A \oplus^{\mathrm{id}} X$ (see* (3.10)*). Then the intersection of all subalgebras of finite codimension in $\mathcal{L}$ is non-zero, while the Frattini radical is trivial*:

$$P_{\mathfrak{S}}(\mathcal{L}) \neq \mathcal{F}(\mathcal{L}) = \{0\}.$$

Proof. If K is a subalgebra of finite codimension in $\mathcal{L}$, then $B = \{a \in A\colon (a;0) \in K\}$ has finite codimension in A. (Indeed, if $a_1, ..., a_n \in A$ are linearly independent modulo B, then $(a_1;0), ..., (a_n;0)$ are linearly independent modulo K.) Hence, since polynomials of T form an infinite-dimensional subspace of A, it follows that B contains a non-zero polynomial $p(T)$.

Similarly, the subspace $M = \{x \in X\colon (0;x) \in K\}$ has finite codimension in X. It is invariant for B because, if $b \in B$, $x \in M$, then $(0;x) \in K$ and $(b;0) \in K$. Hence $(0;bx) = [(b;0),(0;x)] \in K$, so that $bx \in M$. In particular, M is invariant for $p(T)$. By Lemma 7.18(ii), M contains a closed subspace of finite codimension invariant for T. By condition C_2), each such subspace contains the subspace X_ω invariant for T. Hence K contains $\{0\} \oplus^{\mathrm{id}} X_\omega$. Thus $\{0\} \oplus^{\mathrm{id}} X_\omega \subseteq P_{\mathfrak{S}}(\mathcal{L})$.

On the other hand, if A_α is a subspace of finite codimension in A and X_β is a subspace of finite codimension in X invariant for T, then $A_\alpha \oplus^{\mathrm{id}} X$ and $A \oplus^{\mathrm{id}} X_\beta$ are subalgebras of finite codimension in $\mathcal{L}$. As A is commutative, $P_{\mathcal{S}}(A) = \cap A_\alpha = \{0\}$ (see (7.7)). Therefore $P_{\mathfrak{S}}(\mathcal{L}) \subseteq \{0\} \oplus^{\mathrm{id}} X_\omega$. Thus $P_{\mathfrak{S}}(\mathcal{L}) = \{0\} \oplus^{\mathrm{id}} X_\omega$.

As A is commutative, it follows from Example 7.14(i) and (iii) that $\mathcal{F}(\mathcal{L}) = \{0\}$. $\square$

Denote by $\mathrm{Lid}(\mathcal{L})$ the set of all closed Lie ideals of $\mathcal{L}$. We will now construct examples that prove the rest of (7.9).

Example 7.20. (i) $\mathbf{Sem}(P_{\mathfrak{J}}) \subsetneqq \mathbf{Sem}(P_{\mathfrak{S}})$ ([KST1]). Let X be a Banach space and $\dim X = \infty$. Let M be a finite-dimensional Lie subalgebra of $\mathcal{B}(X)$ that has no non-trivial invariant subspaces and let $\mathcal{L} = M \oplus^{\mathrm{id}} X$ (see (3.10)). Let $\mathfrak{Y}$ be the set of all closed subspaces of codimension 1 in X.

For $Y \in \mathfrak{Y}$, $L_Y = \{0\} \oplus^{\mathrm{id}} Y$ is a closed Lie subalgebra of $\mathfrak{L}$ and $\operatorname{codim}(L_Y) = 1 + \dim(M) < \infty$. Thus $\mathcal{L} \in \mathbf{Sem}(P_{\mathfrak{S}})$, as $P_{\mathfrak{S}}(\mathcal{L}) \subseteq \cap_{Y \in \mathfrak{Y}} \mathcal{L}_Y = \{0\}$. Then

$$\operatorname{Lid}(\mathcal{L}) = \{0\} \cup \{J \oplus^{\mathrm{id}} X : J \in \operatorname{Lid}(M)\}.$$

Thus $\{0\} \oplus^{\mathrm{id}} X$ is the smallest non-zero Lie ideal of $\mathcal{L}$. As $\{0\}$ is not a Lie ideal of finite codimension in $\mathcal{L}$, we have $P_{\mathfrak{J}}(\mathcal{L}) = \cap(J \oplus^{\mathrm{id}} X) = \{0\} \oplus^{\mathrm{id}} X$, so that $\mathcal{L} \notin \mathbf{Sem}(P_{\mathfrak{J}})$. Thus $\mathbf{Sem}(P_{\mathfrak{J}}) \subsetneqq \mathbf{Sem}(P_{\mathfrak{S}})$.

In particular, let e be a bounded operator on $X = l_1$ that has no non-trivial closed invariant subspaces (see [R1]). Then the Banach Lie algebra $\mathcal{L} = \mathbb{C}e \oplus^{\mathrm{id}} X$ belongs to $\mathbf{Sem}(P_{\mathfrak{S}})$. As e has no closed invariant subspaces and $\dim(X) = \infty$, $\{0\} \oplus^{\mathrm{id}} X$ is the only proper Lie ideal of $\mathcal{L}$ of finite codimension, so that $\mathcal{L} \notin \mathbf{Sem}(P_{\mathfrak{J}})$.

(ii) $\mathbf{Sem}(P_{\mathfrak{S}^{\max}}) \subsetneqq \mathbf{Sem}(P_{\mathfrak{J}})$. Let $\mathcal{L} = \left\{ a(x,y,z) = \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} : x, y, z \in \mathbb{C} \right\}$ be the Heisenberg 3-dimensional Lie algebra with one-dimensional centre $Z = [\mathcal{L}, \mathcal{L}] = \{a(0,0,z) : z \in \mathbb{C}\}$. As the Lie ideal $\{0\}$ has finite codimension in $\mathcal{L}$, we have $\mathcal{L} \in \mathbf{Sem}(P_{\mathfrak{J}})$. Let M be a maximal Lie subalgebra of $\mathcal{L}$ that does not contain Z. If $\dim(M) = 1$, then $M + Z$ is a 2-dimensional subalgebra larger than M, a contradiction. Thus $\dim(M) = 2$. Hence $\mathcal{L} = M + Z$ and $[M, M] = [M + Z, M + Z] = [\mathcal{L}, \mathcal{L}] = Z$, so that $Z \subseteq M$ — a contradiction. Thus all maximal Lie subalgebras of $\mathcal{L}$ contain Z. Hence $Z \in \cap_{M \in \mathfrak{S}^{\max}} M$, so that $\mathcal{L} \notin \mathbf{Sem}(P_{\mathfrak{S}^{\max}})$. Thus $\mathbf{Sem}(P_{\mathfrak{S}^{\max}}) \subsetneqq \mathbf{Sem}(P_{\mathfrak{J}})$.

(iii) $\mathbf{Sem}(P_{\mathfrak{J}^{\max}}) \subsetneqq \mathbf{Sem}(P_{\mathfrak{S}^{\max}})$. Let $\mathfrak{h}$ be the Lie algebra of all upper triangular matrices in $\mathfrak{sl}(2, \mathbb{C})$, $\mathfrak{n}$ be the Lie subalgebra of $\mathfrak{h}$ of matrices with zero on the diagonal and $\mathfrak{d}$ be the Lie subalgebra of $\mathfrak{h}$ of diagonal matrices. Then $\mathfrak{n}$ is the only maximal Lie ideal of $\mathfrak{h}$, $\mathfrak{n}$ and $\mathfrak{d}$ are maximal Lie subalgebras of $\mathfrak{h}$, so $\mathfrak{n} = P_{\mathfrak{J}^{\max}}(\mathfrak{h}) \neq P_{\mathfrak{S}^{\max}}(\mathfrak{h}) \subseteq \mathfrak{n} \cap \mathfrak{d} = 0$.

To give an example of an infinite-dimensional algebra, we will use the direct product. Let $\mathfrak{h}_i = \mathfrak{h}$ for all $i \in \mathbb{N}$. Then $\widehat{\mathcal{L}} = \widehat{\bigoplus_{i \in \mathbb{N}}} \mathfrak{h}_i = \{a = \{a_i\}_{i \in \mathbb{N}} : \text{all } a_i \in \mathfrak{h} \text{ and } \lim \|a_i\| = 0\}$ — the c_0-direct product of all $\mathfrak{h}_i$ (see (3.11)) is a solvable Banach Lie algebra. For each $i \in \mathbb{N}$, $J_i = \{a \in \widehat{\mathcal{L}}\colon\ a_i \in \mathfrak{n}\}$ is a maximal closed Lie ideal of codimension 1 in $\widehat{\mathcal{L}}$ and $M_i = \{a \in \widehat{\mathcal{L}}\colon\ a_i \in \mathfrak{d}\}$ is a maximal closed Lie subalgebra in $\widehat{\mathcal{L}}$ of codimension 1. If $J \in \mathfrak{J}^{\max}_{\widehat{\mathcal{L}}}$ then $[\mathfrak{h}_i, J]$ is either $\{0\}$ or $\mathfrak{n}_i$ for each $i \in \mathbb{N}$. If $[\mathfrak{h}_i, J] = \{0\}$ then $J + \mathfrak{n}_i$ is a closed proper Lie ideal of $\widehat{\mathcal{L}}$ larger than J. This contradiction shows that $[\mathfrak{h}_i, J] = \mathfrak{n}_i$ for each $i \in \mathbb{N}$. Hence either $\mathfrak{h}_i \subseteq J$ or $\mathfrak{n}_i \subseteq J$. As $\widehat{\mathcal{L}}$ is the c_0-direct product of all $\mathfrak{h}_i$, we conclude that J coincides with one of J_i. Thus $\mathfrak{J}^{\max}_{\widehat{\mathcal{L}}} = \{J_i\colon\ i \in \mathbb{N}\}$. Therefore $\mathcal{L} \notin \mathbf{Sem}(P_{\mathfrak{J}^{\max}})$ and $\mathcal{L} \in \mathbf{Sem}(P_{\mathfrak{S}^{\max}})$, as

$$P_{\mathfrak{J}^{\max}}(\mathcal{L}) = \cap_{i \in \mathbb{N}} J_i \neq \{0\} \text{and } P_{\mathfrak{S}^{\max}}(\mathcal{L}) = (\cap_{i \in \mathbb{N}} J_i) \cap (\cap_{i \in \mathbb{N}} M_i) = \{0\}.$$

8. Frattini-semisimple Banach Lie algebras

As in the classical theory of finite-dimensional Lie algebras, the most "tractable" infinite dimensional Banach Lie algebras are Frattini-semisimple Lie algebras. In this section we show that they admit chains decreasing to $\{0\}$ of closed Lie subalgebras and even Lie 2-step subideals with finite-dimensional quotients. We also consider a subclass of $\mathbf{Sem}(\mathcal{F})$ that consists of Banach Lie algebras that admit chains of closed Lie ideals decreasing to $\{0\}$ with finite-dimensional quotients. We call them *strongly Frattini-semisimple* and prove that these algebras can be equivalently defined in terms of the structure of the sets of their Lie ideals, of their $\mathcal{F}$-primitive Lie ideals and of their commutative Lie ideals.

8.1. **Chains of Lie subalgebras and ideals in Banach Lie algebras.** We begin with a result which, in particular, shows that separable $P_{\mathfrak{S}}$- and $P_{\mathfrak{J}}$-semisimple Banach Lie algebras are characterized by the existence of sequences of Lie subalgebras and Lie ideals of finite codimension, respectively, that decrease to $\{0\}$. Recall that $P_{\mathfrak{S}}(\mathcal{L}) \subseteq P_{\mathfrak{J}}(\mathcal{L})$.

Proposition 8.1. (i) *Each Banach Lie algebra $\mathcal{L}$ has complete, lower finite-gap chains of closed Lie ideals between $\mathcal{L}$ and $P_{\mathfrak{J}}(\mathcal{L})$, and of closed Lie subalgebras between $\mathcal{L}$ and $P_{\mathfrak{S}}(\mathcal{L})$.*

(ii) *Each separable Banach Lie algebra $\mathcal{L}$ has a sequence $\{J_n\}_{n=1}^{\infty}$ of closed Lie ideals of finite codimension and a sequence $\{L_n\}_{n=1}^{\infty}$ of closed Lie subalgebras of finite codimension such that*

$$J_{n+1} \subseteq J_n \text{ and } \cap_{n=0}^{\infty} J_n = P_{\mathfrak{J}}(\mathcal{L}); \quad L_{n+1} \subseteq L_n \text{ and } \cap_{n=0}^{\infty} L_n = P_{\mathfrak{S}}(\mathcal{L}).$$

Proof. (i) The family $\mathfrak{J}_{\mathcal{L}}$ consists of all closed Lie ideals of finite codimension in $\mathcal{L}$. Hence its $\mathfrak{p}$-completion $\mathfrak{J}_{\mathcal{L}}^{\mathfrak{p}}$ — the set of intersections of Lie ideals from all subfamilies of $\mathfrak{J}_{\mathcal{L}}$ — consists of Lie ideals of $\mathcal{L}$. The family $\mathfrak{S}_{\mathcal{L}}$ consists of all closed Lie subalgebras of finite codimension in $\mathcal{L}$ and its $\mathfrak{p}$-completion $\mathfrak{S}_{\mathcal{L}}^{\mathfrak{p}}$ consists of Lie subalgebras of $\mathcal{L}$. By Lemma 6.3, $\mathfrak{J}_{\mathcal{L}}^{\mathfrak{p}}$ and $\mathfrak{S}_{\mathcal{L}}^{\mathfrak{p}}$ are lower finite-gap families. Thus (i) follows from Theorem 6.6.

Part (ii) follows from Theorem 6.10. $\square$

Corollary 8.2. *A separable Banach Lie algebra is $P_{\mathfrak{S}}$-semisimple if and only if it has a chain $\{L_n\}_{n=1}^{\infty}$ of closed Lie subalgebras of finite codimension such that $L_{n+1} \subseteq L_n$ and $\cap_{n=0}^{\infty} L_n = \{0\}$.*

It is $P_{\mathfrak{J}}$-semisimple if and only if it has a chain $\{J_n\}_{n=1}^{\infty}$ of closed Lie ideals of finite codimension such that $J_{n+1} \subseteq J_n$ and $\cap_{n=0}^{\infty} J_n = \{0\}$.

Proposition 8.3. *Let the quotient Lie algebra $\mathcal{L}/P_{\mathfrak{S}}(\mathcal{L})$ have no characteristic commutative, infinite-dimensional Lie ideals. Then $\mathcal{L}$ has a maximal lower finite-gap chain of closed characteristic Lie ideals between $\mathcal{L}$ and $P_{\mathfrak{S}}(\mathcal{L})$.*

Proof. Recall that $P_{\mathfrak{S}}(\mathcal{L}) = \cap_{L \in \mathfrak{S}_{\mathcal{L}}} L$ is a characteristic Lie ideal of $\mathcal{L}$. Firstly assume that $P_{\mathfrak{S}}(\mathcal{L}) = \{0\}$. Let G be the family of all closed characteristic Lie ideals of $\mathcal{L}$. Then $\mathfrak{p}(G) = \{0\} \in G$ and $\mathfrak{s}(G) = \mathcal{L} \in G$. For each subfamily G' of G, the Lie ideal $\mathfrak{p}(G') = \cap_{J \in G'} J$ of $\mathcal{L}$ is characteristic. Hence G is $\mathfrak{p}$-complete.

Let $\{0\} \neq I \in G$. If $\dim I < \infty$ then $\{0\}$ has finite codimension in I. Let $\dim I = \infty$. As $\cap_{L \in \mathfrak{S}_{\mathcal{L}}} L = \{0\}$, there is $L \in \mathfrak{S}_{\mathcal{L}}$ that does not contain I. By Lemma 6.1, $I \cap L$ is a proper closed Lie subalgebra of finite codimension in I. By our assumption, I is non-commutative. Hence, by Theorem 2.5(ii), I has a proper closed characteristic Lie ideal K of finite codimension. Then, by Lemma 2.4(ii), $K \lhd^{\mathrm{ch}} \mathcal{L}$, so that $K \in G$ and $0 < \dim(I/K) < \infty$. Hence G is a $\mathfrak{p}$-complete lower finite-gap family. We have from Lemma 6.5 that G has a maximal, lower finite-gap chain C of closed characteristic Lie ideals such that $\mathfrak{p}(C) = \{0\}$ and $\mathfrak{s}(C) = \mathcal{L}$.

Let now $P_{\mathfrak{S}}(\mathcal{L}) \neq \{0\}$ and $\dim(\mathcal{L}/P_{\mathfrak{S}}(\mathcal{L})) = \infty$. Set $\widetilde{\mathcal{L}} = \mathcal{L}/P_{\mathfrak{S}}(\mathcal{L})$. As $P_{\mathfrak{S}}$ is upper stable (see Theorem 7.3), we have from (3.4) that $P_{\mathfrak{S}}(\widetilde{\mathcal{L}}) = \{0\}$. By our assumption, $\widetilde{\mathcal{L}}$ has no infinite-dimensional commutative characteristic Lie ideals. Hence, by the above, $\widetilde{\mathcal{L}}$ has a maximal, lower finite-gap chain $\widetilde{C} = \{\widetilde{I_\lambda}\}$ of closed characteristic Lie ideals such that $\mathfrak{p}(\widetilde{C}) = \{0\}$ and $\mathfrak{s}(\widetilde{C}) = \widetilde{\mathcal{L}}$. By Lemma 2.3, the preimages I_λ of $\widetilde{I_\lambda}$ in $\mathcal{L}$ are closed characteristic Lie ideals of $\mathcal{L}$. Hence $C = \{I_\lambda\}$ is a maximal, lower finite-gap chain of closed characteristic Lie ideals of $\mathcal{L}$ with $\mathfrak{p}(C) = P_{\mathfrak{S}}(\mathcal{L})$ and $\mathfrak{s}(C) = \mathcal{L}$. $\square$

We will prove in this subsection that $\mathcal{F}$-semisimple algebras are characterized by the existence of complete lower finite-gap chains of closed Lie subalgebras decreasing to $\{0\}$. Since the radical $\mathcal{F}$ is generated by the preradical $P_{\mathfrak{J}}$, one can expect that the same holds for chains of Lie ideals. However, this is not true in general; the class of algebras for which this is true will be considered in the next subsection. Nevertheless, we will see that symmetry can be partially recovered if instead of ideals one works with 2-step subideals.

Recall (see Definition 2.7) that a Lie subalgebra I of $\mathcal{L}$ is a 2-*step Lie subideal* if it is a Lie ideal of some Lie ideal of $\mathcal{L}$. We write $I \lhd_2 \mathcal{L}$ if I is closed. This matches the notation in Definition 2.7 because a closed 2-step Lie subideal of $\mathcal{L}$ is clearly a Lie ideal of a closed Lie ideal of $\mathcal{L}$. Clearly, all Lie ideals are 2-step Lie subideals.

Lemma 8.4. *Let $\mathcal{L}$ be a Banach Lie algebra. Then*

(i) *the sum of a Lie ideal of $\mathcal{L}$ and a 2-step Lie subideal of $\mathcal{L}$ is a 2-step Lie subideal of $\mathcal{L}$;*
(ii) *the intersection of a family of 2-step Lie subideals of $\mathcal{L}$ is a 2-step Lie subideal of $\mathcal{L}$.*

Proof. (i) If I is a Lie ideal of some Lie ideal K of $\mathcal{L}$ and $J \lhd \mathcal{L}$, then $I+J$ is a Lie ideal of $K+J$ and $K+J$ is a Lie ideal of $\mathcal{L}$.

(ii) Let I_α be a Lie ideal of some Lie ideal K_α of $\mathcal{L}$ for each $\alpha \in \Lambda$, where Λ is an index set. Then $\cap_\alpha K_\alpha$ is a Lie ideal of $\mathcal{L}$ and $\cap_\alpha I_\alpha$ is a Lie ideal of $\cap_\alpha K_\alpha$. $\square$

Proposition 8.5. *For each Banach Lie algebra $\mathcal{L}$, there is a complete, lower finite-gap chain of closed Lie 2-step subideals of $\mathcal{L}$ between $\mathcal{F}(\mathcal{L})$ and $\mathcal{L}$.*

Proof. Let $\left\{P^{\alpha}_{\mathfrak{J}}(\mathcal{L})\right\}_{\alpha=0}^{\beta}$ be the $P_{\mathfrak{J}}$-superposition series of closed Lie ideals of $\mathcal{L}$. Then $P^{\beta}_{\mathfrak{J}}(\mathcal{L}) = P^{\circ}_{\mathfrak{J}}(\mathcal{L}) = \mathcal{F}(\mathcal{L})$. As $P^{\alpha+1}_{\mathfrak{J}}(\mathcal{L}) = P_{\mathfrak{J}}\left(P^{\alpha}_{\mathfrak{J}}(\mathcal{L})\right)$, we have from Proposition 8.1 that there is a complete chain C_α of closed Lie ideals of $P^{\alpha}_{\mathfrak{J}}(\mathcal{L})$ such that it is a lower finite-gap chain, $\mathfrak{s}(C_\alpha) = P^{\alpha}_{\mathfrak{J}}(\mathcal{L})$ and $\mathfrak{p}(C_\alpha) = P^{\alpha+1}_{\mathfrak{J}}(\mathcal{L})$. As $P^{\alpha}_{\mathfrak{J}}(\mathcal{L}) \lhd \mathcal{L}$, these Lie ideals are Lie 2-step subideals of $\mathcal{L}$. Hence $\left(\cup_{\alpha=0}^{\beta} C_\alpha\right) \cup \left(\cup_{\alpha=0}^{\beta} P^{\alpha}_{\mathfrak{J}}(\mathcal{L})\right)$ is a complete, lower finite-gap chain between $\mathcal{F}(\mathcal{L})$ and $\mathcal{L}$ that consists of closed 2-step Lie subideals of $\mathcal{L}$. $\square$

The following theorem describes $\mathcal{F}$-semisimple (Frattini-semisimple) Lie algebras in terms of lower finite-gap chains of Lie subalgebras and 2-step ideals.

Theorem 8.6. *Let $\mathcal{L}$ be a Banach Lie algebra. Then the following conditions are equivalent:*

(i) *$\mathcal{L}$ is $\mathcal{F}$-semisimple.*

(ii) *$\mathcal{L}$ has a complete, lower finite-gap chain of closed Lie 2-step subideals from $\{0\}$ to $\mathcal{L}$.*

(iii) *$\mathcal{L}$ has a complete, lower finite-gap chain of closed Lie subalgebras from $\{0\}$ to $\mathcal{L}$.*

(iv) *The set of all closed Lie 2-step subideals of $\mathcal{L}$ is a $\mathfrak{p}$-complete, lower finite-gap family.*

(v) *The set of all closed Lie subalgebras of $\mathcal{L}$ is a $\mathfrak{p}$-complete, lower finite-gap family.*

Proof. (i) $\Longrightarrow$ (ii) follows from Proposition 8.5.

(ii) $\Longrightarrow$ (iii) is obvious. The set in (v) is $\mathfrak{p}$-complete. The set in (iv) contains $\mathcal{L}$, so that it is $\mathfrak{p}$-complete by Lemma 8.4(ii). Hence (iv) $\Longleftrightarrow$ (ii) and (v) $\Longleftrightarrow$ (iii) follow from Theorem 6.6.

(iii) $\Longrightarrow$ (i) By Lemma 6.5, the chain $C = \{M_\alpha\}_{\alpha=0}^{\beta}$ of closed Lie subalgebras in (iii) is a strictly decreasing transfinite chain with $M_0 = \mathcal{L}$ and $M_\beta = \{0\}$. Let $\mathcal{F}(\mathcal{L}) \neq \{0\}$ and α_0 be the first ordinal such that M_{α_0} does not contain $\mathcal{F}(\mathcal{L})$. Clearly, α_0 is not a limit ordinal. Hence $\alpha_0 = \delta+1$ and M_{α_0} is a Lie subalgebra of finite codimension in M_δ. Thus $\mathcal{F}(M_\delta) \subseteq P_{\mathfrak{S}}(M_\delta) \subseteq M_{\alpha_0}$. As $\mathcal{F}(\mathcal{L})$ is a Lie ideal of M_δ and $\mathcal{F}$ is a radical, we have $\mathcal{F}(\mathcal{L}) = \mathcal{F}(\mathcal{F}(\mathcal{L})) \subseteq \mathcal{F}(M_\delta) \subset M_{\alpha_0}$, a contradiction. Thus all M_α contain $\mathcal{F}(\mathcal{L})$, whence $\mathcal{F}(\mathcal{L}) = \{0\}$. $\square$

A closed ideal I of a Banach Lie algebra $\mathcal{L}$ is *$\mathcal{F}$-primitive* (*Frattini-primitive*) if $\mathcal{L}/I$ is $\mathcal{F}$-semisimple (cf. Definition 3.9). We saw in Theorem 7.12(ii) that $\mathcal{D}(\mathcal{L})$ is $\mathcal{F}$-primitive. Denote the set of all $\mathcal{F}$-primitive ideals of $\mathcal{L}$ by $\mathrm{Prim}_{\mathcal{F}}(\mathcal{L})$.

Lemma 8.7. (i) *A closed Lie ideal I of $\mathcal{L}$ is $\mathcal{F}$-primitive if and only if there is a complete, lower finite-gap chain of closed Lie subalgebras between I and $\mathcal{L}$.*

(ii) *Let $I \in \mathrm{Prim}_{\mathcal{F}}(\mathcal{L})$. If $J \lhd \mathcal{L}$, $J \subseteq I$ and $\dim(I/J) < \infty$, then $J \in \mathrm{Prim}_{\mathcal{F}}(\mathcal{L})$.*

(iii) *Each complete, lower finite-gap chain C of closed Lie ideals of $\mathcal{L}$ with $\mathfrak{s}(C) = \mathcal{L}$ consists of $\mathcal{F}$-primitive ideals of $\mathcal{L}$.*

(iv) *The set $\mathrm{Prim}_{\mathcal{F}}(\mathcal{L})$ is $\mathfrak{p}$-complete, $\mathfrak{s}(\mathrm{Prim}_{\mathcal{F}}(\mathcal{L})) = \mathcal{L}$ and $\mathfrak{p}(\mathrm{Prim}_{\mathcal{F}}(\mathcal{L})) = \mathcal{F}(\mathcal{L})$.*

(v) *Let $\mathcal{M}$ be a closed Lie subalgebra of $\mathcal{L}$. If $I \in \mathrm{Prim}_{\mathcal{F}}(\mathcal{L})$ then $I \cap \mathcal{M} \in \mathrm{Prim}_{\mathcal{F}}(\mathcal{M})$.*

Proof. (i) If $\mathcal{L}/I$ is $\mathcal{F}$-semisimple then, by Theorem 8.6, there is a complete, lower finite-gap chain of closed Lie subalgebras of $\mathcal{L}/I$ between $\{0\}$ and $\mathcal{L}/I$. Their preimages in $\mathcal{L}$ form a complete, lower finite-gap chain of closed Lie subalgebras of $\mathcal{L}$ between I and $\mathcal{L}$.

Conversely, if $C = \{L_\lambda\}$ is such a chain, then the quotients L_λ/I form a complete, lower finite-gap chain of closed Lie subalgebras of $\mathcal{L}/I$ between $\{0\}$ and $\mathcal{L}/I$. Thus $\mathcal{L}/I$ is $\mathcal{F}$-semisimple.

(ii) By (i), there is a complete, lower finite-gap chain C of closed Lie subalgebras between I and $\mathcal{L}$. Then $C' = J \cup C$ is the same type of chain between J and $\mathcal{L}$. By (i), J is $\mathcal{F}$-primitive.

(iii) For $I \in C$, $\{J \in C\colon I \subseteq J\}$ is a complete, lower finite-gap chain. By (i), I is $\mathcal{F}$-primitive.

(iv) As $\mathcal{L} \in \mathrm{Prim}_{\mathcal{F}}(\mathcal{L})$, we have $\mathfrak{s}(\mathrm{Prim}_{\mathcal{F}}(\mathcal{L})) = \mathcal{L}$. The rest follows from Theorem 3.10.

(v) If $I \in \mathrm{Prim}_{\mathcal{F}}(\mathcal{L})$ then, by (i), $\mathcal{L}$ has a complete, lower finite-gap chain C of closed Lie subalgebras between I and $\mathcal{L}$. By Corollary 6.7, $C_{\mathcal{M}} = \{J \cap \mathcal{M}\colon\ J \in C\}$ is a complete, lower finite-gap chain of closed Lie subalgebras of $\mathcal{M}$ between $I \cap \mathcal{M}$ and $\mathcal{M}$. Hence, by (i), $I \cap \mathcal{M} \in \mathrm{Prim}_{\mathcal{F}}(\mathcal{M})$. $\square$

Let G be the set of all closed Lie subalgebras of $\mathcal{L}$. It is $\mathfrak{p}$-complete. Comparing (6.1), Theorem 6.11 and Lemma 8.7, we have that $G_{\mathrm{f}} = \mathrm{Prim}_{\mathcal{F}}(\mathcal{L})$ and $\Delta_G = \mathcal{F}(\mathcal{L})$. This and Lemma 6.5 yield

Corollary 8.8. (i) *$\mathcal{F}(\mathcal{L}) = \mathfrak{p}(C)$ for each maximal, lower finite-gap chain C of closed Lie subalgebras of $\mathcal{L}$ with $\mathfrak{s}(C) = \mathcal{L}$.*

(ii) *Each $\mathfrak{p}$-complete, lower finite-gap chain C of closed Lie subalgebras of $\mathcal{L}$ with $\mathfrak{s}(C) = \mathcal{L}$ extends to a maximal, lower finite-gap chain of closed Lie subalgebras.*

In general, for a radical R, a subalgebra of an R-semisimple Lie algebra is not necessarily R-semisimple. However, for the Frattini radical $\mathcal{F}$, the situation is much better.

Corollary 8.9. *If $\mathcal{L} \in \mathbf{Sem}(\mathcal{F})$ then each closed Lie subalgebra $\mathcal{M}$ of $\mathcal{L}$ is $\mathcal{F}$-semisimple.*

Proof. As $\{0\} \in \mathrm{Prim}_{\mathcal{F}}(\mathcal{L})$, by Lemma 8.7(v), $\{0\} \in \mathrm{Prim}_{\mathcal{F}}(\mathcal{M})$. Hence $\mathcal{M} \in \mathbf{Sem}(\mathcal{F})$. $\square$

Now we consider the sets of $\mathcal{F}$-primitive characteristic Lie ideals in Banach Lie algebras.

Theorem 8.10. *Let $\mathcal{L}$ be a Banach Lie algebra.*

(i) *$\mathcal{L}$ has a maximal chain of $\mathcal{F}$-primitive ideals between $\mathcal{F}(\mathcal{L})$ and $\mathcal{L}$.*

(ii) *$\mathcal{L}$ has a maximal lower finite-gap chain of $\mathcal{F}$-primitive Lie ideals between $P_{\mathfrak{J}}(\mathcal{L})$ and $\mathcal{L}$.*

(iii) *Let R be one of the preradicals $P_{\mathfrak{S}}$, $P_{\mathfrak{J}}$, $P_{\mathfrak{S}^{\max}}$, $P_{\mathfrak{J}^{\max}}$. Then*

a) *for each $\mathcal{F}$-primitive Lie ideal I of $\mathcal{L}$, the Lie ideal $P_R(I)$ is also $\mathcal{F}$-primitive;*

b) *the characteristic Lie ideals $P_R^{\alpha}(\mathcal{L})$ in the R-superposition series are $\mathcal{F}$-primitive.*

(iv) *Let $\mathcal{L}/P_{\mathfrak{S}}(\mathcal{L})$ have no characteristic commutative, infinite-dimensional Lie ideals. Then $\mathcal{L}$ has a maximal, lower finite-gap chain of $\mathcal{F}$-primitive characteristic Lie ideals between $P_{\mathfrak{S}}(\mathcal{L})$ and $\mathcal{L}$.*

Proof. (i) follows from Lemmas 6.5 and 8.7(iv).

(ii) follows from Proposition 8.1(i) and Lemma 8.7(iii).

(iii) a) By Lemma 2.4(i), $P_R(I)$ is a Lie ideal of $\mathcal{L}$. As I is $\mathcal{F}$-primitive, we have from Lemma 8.7(i) that there is a complete, lower finite-gap chain C_I of closed Lie subalgebras between I and $\mathcal{L}$. By Proposition 8.1(i), there is a complete lower finite-gap chain C' of closed Lie subalgebras between $R(I)$ and I. Hence $C = C_I \cup C'$ is a complete lower finite-gap chain of closed Lie subalgebras between $R(I)$ and $\mathcal{L}$. Thus, by Lemma 8.7(i), $R(I)$ is $\mathcal{F}$-primitive.

(iii) b) follows by induction. Let $P_R^{\alpha}(\mathcal{L})$ be $\mathcal{F}$-primitive. By (i), $P_R^{\alpha+1}(\mathcal{L})$ is also $\mathcal{F}$-primitive. The case of a limit ordinal α follows from Lemma 8.7(iv).

(iv) follows from Proposition 8.1(i) and Lemma 8.7(iii). $\square$

Denote, as in Example 7.20, by $\mathrm{Lid}(\mathcal{L})$ the lattice of all closed Lie ideals in $\mathcal{L}$. Denote by

1) $\mathcal{A}_{\mathcal{L}}$ the set of all closed commutative Lie ideals of $\mathcal{L}$;

2) $\mathcal{A}_{\mathcal{L}}^{\mathrm{ch}}$ the set of all characteristic Lie ideals of $\mathcal{L}$ in $\mathcal{A}_{\mathcal{L}}$;

3) $\mathcal{A}_{\mathcal{L}}^{\mathrm{Prim}} := \mathcal{A}_{\mathcal{L}} \cap \mathrm{Prim}_{\mathcal{F}}(\mathcal{L})$ the set of all $\mathcal{F}$-primitive Lie ideals of $\mathcal{L}$ in $\mathcal{A}_{\mathcal{L}}$.

Proposition 8.11. *Let $\mathcal{L} \in \mathbf{Sem}(\mathcal{F})$. Then*

(i) *the set $\mathrm{Lid}(\mathcal{L}) \diagdown \mathcal{A}_{\mathcal{L}}$ is a lower finite-gap family modulo $\mathcal{A}_{\mathcal{L}}$ (see Definition 6.13);*

(ii) *the set of all infinite-dimensional non-commutative closed characteristic Lie ideals of $\mathcal{L}$ is a lower finite-gap family modulo $\mathcal{A}_{\mathcal{L}}^{\mathrm{ch}}$;*

(iii) *the set $\mathrm{Prim}_{\mathcal{F}}(\mathcal{L}) \diagdown \mathcal{A}_{\mathcal{L}}^{\mathrm{Prim}}$ is a lower finite-gap family modulo $\mathcal{A}_{\mathcal{L}}^{\mathrm{Prim}}$.*

Proof. Let J be a non-commutative Lie ideal of $\mathcal{L}$ and $\dim J = \infty$. By Theorem 8.6(v), J has a proper subalgebra of finite codimension. By Corollary 2.6, it contains a closed Lie ideal I of $\mathcal{L}$ that has non-zero finite codimension in J. Part (i) is proved.

If J is characteristic then, by Corollary 2.6, I is also characteristic. This proves (ii).

Let $J \in \operatorname{Prim}_{\mathcal{F}}(\mathcal{L}) \diagdown \mathcal{A}_{\mathcal{L}}$. Then, by (i), $\mathcal{L}$ has a closed Lie ideal I such that $I \subsetneqq J$ and $\dim(J/I) < \infty$. By Lemma 8.7(ii), $I \in \operatorname{Prim}_{\mathcal{F}}(\mathcal{L}) \subseteq (\operatorname{Prim}_{\mathcal{F}}(\mathcal{L}) \diagdown \mathcal{A}_{\mathcal{L}}^{\mathrm{Prim}}) \cup \mathcal{A}_{\mathcal{L}}^{\mathrm{Prim}}$. $\square$

8.2. **Strongly Frattini-semisimple Banach Lie algebras.** Theorem 8.6 gives us a satisfactory description of $\mathcal{F}$-semisimple Lie algebras in terms of lower finite-gap chains of Lie subalgebras and 2-step subideals. These algebras may also have lower finite-gap chains of Lie ideals. However, there are $\mathcal{F}$-semisimple algebras where these chains do not stretch from $\mathcal{L}$ to $\{0\}$.

Indeed, the Lie algebra $\mathcal{L} = M \oplus^{\mathrm{id}} X$ in Example 7.20(i) is $\mathcal{F}$-semisimple (see Example 7.14(iii)) and its Lie ideal $J = \{0\} \oplus^{\mathrm{id}} X$ is infinite-dimensional, commutative and contained in each non-zero Lie ideal of $\mathcal{L}$. Hence if C is a maximal lower finite-gap chains of Lie ideals of $\mathcal{L}$ then $\mathfrak{p}(C) = J$. Thus C does not continue to $\{0\}$.

Definition 8.12. *A Banach Lie algebra $\mathcal{L}$ is strongly $\mathcal{F}$-semisimple* (*Frattini-semisimple*) *if there is a complete, lower finite-gap chain of closed Lie ideals of $\mathcal{L}$ between $\{0\}$ and $\mathcal{L}$.*

We will see later that each $\mathcal{F}$-semisimple Banach Lie algebra $\mathcal{L}$ contains a characteristic commutative Lie ideal J such that $\mathcal{L}/J$ is strongly $\mathcal{F}$-semisimple. Therefore, for Lie algebras without commutative Lie ideals, these two notions coincide. Thus the presence of the commutative Lie ideal $J = \{0\} \oplus^{\mathrm{id}} X$ in the above example is not incidental.

The following result shows that one can define strongly $\mathcal{F}$-semisimple Lie algebras as algebras with complete, lower finite-gap chains of *$\mathcal{F}$-primitive* Lie ideals between $\{0\}$ and $\mathcal{L}$.

Theorem 8.13. *Let $\mathcal{L}$ be a Banach Lie algebra. Then the following conditions are equivalent.*

(i) *$\mathcal{L}$ is strongly $\mathcal{F}$-semisimple.*
(ii) *$\mathcal{L}$ has a complete, lower finite-gap chain of $\mathcal{F}$-primitive ideals between $\{0\}$ and $\mathcal{L}$.*
(iii) *The set $\operatorname{Lid}(\mathcal{L})$ of all closed Lie ideals of $\mathcal{L}$ is a lower finite-gap family.*
(iv) *The set $\operatorname{Prim}_{\mathcal{F}}(\mathcal{L})$ is a lower finite-gap family containing $\{0\}$.*

Proof. The set $\operatorname{Lid}(\mathcal{L})$ is $\mathfrak{p}$-complete. The set $\operatorname{Prim}_{\mathcal{F}}(\mathcal{L})$ is $\mathfrak{p}$-complete by Lemma 8.7(iv); (i) $\Longrightarrow$ (ii) follows from Lemma 8.7(iii); (ii) $\Longrightarrow$ (i) is obvious; (iii) $\Longleftrightarrow$ (i) and (iv) $\Longleftrightarrow$ (ii) follow from Theorem 6.6. $\square$

Strongly Frattini-semisimple algebras can be characterized in the class of Frattini-semisimple algebras by the structure of the sets of their commutative ideals.

Theorem 8.14. *Let $\mathcal{L} \in \mathfrak{L}$. Then the following conditions are equivalent.*

(i) *$\mathcal{L}$ is strongly $\mathcal{F}$-semisimple.*
(ii) *The set $\mathcal{A}_{\mathcal{L}}$ of all closed commutative Lie ideals of $\mathcal{L}$ is a lower finite-gap family.*
(iii) *The set $\mathcal{A}_{\mathcal{L}}^{\mathrm{Prim}} = \mathcal{A}_{\mathcal{L}} \cap \operatorname{Prim}_{\mathcal{F}}(\mathcal{L})$ is a lower finite-gap family.*
(iv) *$\mathcal{L}$ has a complete, lower finite-gap chain of closed Lie ideals between $\{0\}$ and $\mathfrak{s}(\mathcal{A}_{\mathcal{L}})$.*
(v) *$\mathcal{L}$ has a complete, lower finite-gap chain of closed ideals between $\{0\}$ and $\mathfrak{s}\left(\mathcal{A}_{\mathcal{L}}^{\mathrm{Prim}}\right)$.*

Proof. (i) $\Longrightarrow$ (ii) follows from Theorem 8.13(iii), and (ii) $\Longrightarrow$ (iii) follows from Lemma 8.7(ii).

(iii) $\Longrightarrow$ (i). It follows from Lemma 6.14 and Proposition 8.11 that $\operatorname{Prim}_{\mathcal{F}}(\mathcal{L})$ is a lower finite-gap family. By Theorem 8.13, $\mathcal{L}$ is strongly $\mathcal{F}$-semisimple.

(i) $\Longrightarrow$ (iv) and (v). By Theorem 8.13(iii), $\operatorname{Lid}(\mathcal{L})$ is a $\mathfrak{p}$-complete, lower finite-gap family. Hence, for any $J \in \operatorname{Lid}(\mathcal{L})$, the set $\operatorname{Lid}_J(\mathcal{L}) = \{I \in \operatorname{Lid}(\mathcal{L})\colon\ I \subseteq J\}$ is a $\mathfrak{p}$-complete, lower finite-gap family. Thus (iv) and (v) follow from Lemma 6.5.

(iv) $\Longrightarrow$ (ii). $\operatorname{Lid}_{\mathfrak{s}(\mathcal{A}_{\mathcal{L}})}(\mathcal{L})$ is a $\mathfrak{p}$-complete family. If the required chain exists then, by Theorem 6.6, $\operatorname{Lid}_{\mathfrak{s}(\mathcal{A}_{\mathcal{L}})}(\mathcal{L})$ is a lower finite-gap family. As $\mathcal{A}_{\mathcal{L}} \subseteq \operatorname{Lid}_{\mathfrak{s}(\mathcal{A}_{\mathcal{L}})}(\mathcal{L})$, we easily have that $\mathcal{A}_{\mathcal{L}}$ is a lower finite-gap family.

(v) $\Longrightarrow$ (iii). Replacing $\mathfrak{s}(\mathcal{A}_{\mathcal{L}})$ by $\mathfrak{s}\left(\mathcal{A}_{\mathcal{L}}^{\mathrm{Prim}}\right)$ in (iv) $\Longrightarrow$ (ii) and using Lemma 8.7(ii), we obtain that $\mathcal{A}_{\mathcal{L}}^{\mathrm{Prim}}$ is a lower finite-gap family. $\square$

Corollary 8.15. *Let $\mathcal{L} \in$ $\mathbf{Sem}(\mathcal{F})$. If the set $\mathcal{A}_{\mathcal{L}}^{\mathrm{ch}}$ of all closed commutative characteristic Lie ideals of $\mathcal{L}$ is a lower finite-gap family, then $\mathcal{L}$ has a maximal, lower finite-gap chain of characteristic Lie ideals between $\{0\}$ and $\mathcal{L}$.*

Proof. If $\mathcal{A}_{\mathcal{L}}^{\mathrm{ch}}$ is a lower finite-gap family, we have from Lemma 6.14 and Proposition 8.11 that the set $\mathrm{Lid}^{\mathrm{ch}}(\mathcal{L})$ of all closed characteristic Lie ideals of $\mathcal{L}$ is a lower finite-gap family. As $\mathcal{L},\{0\} \in \mathrm{Lid}^{\mathrm{ch}}(\mathcal{L})$ and the intersection of any subfamily of characteristic Lie ideals is also a characteristic Lie ideal, $\mathrm{Lid}^{\mathrm{ch}}(\mathcal{L})$ is $\mathfrak{p}$-complete. Applying Lemma 6.5, we complete the proof. □

Let G be a family of closed subspaces in a Banach space X. A subspace $Y \in G$ is called *lower essential in* G if the set

$$G_{-}(Y) = \{Z \in G : Z \subsetneqq Y\} \neq \varnothing \text{ and } \dim(Y/Z) = \infty \text{ for each } Z \in G_{-}(Y).$$

Denote by $\mathrm{Ess}_l(G)$ the set of all lower essential subspaces Y in G.

Corollary 8.16. *$\mathcal{L}$ is strongly $\mathcal{F}$-semisimple if and only if $\mathcal{A}_{\mathcal{L}}^{\mathrm{Prim}} \cap \mathrm{Ess}_l(\mathcal{A}_{\mathcal{L}}) = \varnothing$.*

Proof. If $\mathcal{A}_{\mathcal{L}}^{\mathrm{Prim}} \cap \mathrm{Ess}_l(\mathcal{A}_{\mathcal{L}}) \neq \{0\}$ then $\mathcal{A}_{\mathcal{L}}$ is not a lower finite-gap family. By Theorem 8.14, $\mathcal{L}$ is not strongly $\mathcal{F}$-semisimple. Conversely, if $\mathcal{A}_{\mathcal{L}}^{\mathrm{Prim}} \cap \mathrm{Ess}_l(\mathcal{A}_{\mathcal{L}}) = \varnothing$ then, for each $Y \in \mathcal{A}_{\mathcal{L}}^{\mathrm{Prim}}$, $Y \neq \mathfrak{p}(\mathcal{A}_{\mathcal{L}})$, there is $Z \in (\mathcal{A}_{\mathcal{L}})_{-}(Y)$ of finite codimension in Y. By Lemma 8.7(ii), $Z \in \mathcal{A}_{\mathcal{L}}^{\mathrm{Prim}}$. Hence $\mathcal{A}_{\mathcal{L}}^{\mathrm{Prim}}$ is a lower finite-gap family. By Theorem 8.14, $\mathcal{L}$ is strongly $\mathcal{F}$-semisimple. □

In two examples below H is a Hilbert space with an orthonormal basis $\{e_i\}_{i=1}^{\infty}$, and $H_0 = \{0\}$ and $H_n = \sum_{i=1}^{n} \oplus \mathbb{C}e_i$, for $n > 0$, are its finite-dimensional subspaces. In the first example we consider a $\mathcal{D}$-semisimple (hence $\mathcal{F}$-semisimple) Banach Lie algebra $\mathcal{L}$ that has a commutative $\mathcal{F}$-primitive ideal in $\mathrm{Ess}_l(\mathcal{A}_{\mathcal{L}})$, so it is not strongly $\mathcal{F}$-semisimple by Corollary 8.16.

Example 8.17. Consider the nest $G = H \cup \{H_n\}_{n=0}^{\infty}$ — a complete chain of subspaces from $\{0\}$ to H. Let $M = \mathrm{Alg}(G)$ be the algebra of all operators in $B(H)$ leaving each subspace from G invariant. Then M has a chain of closed two-sided ideals $I_n = \{T \in M : T|_{H_n} = 0\}$ that have finite codimension in M and $\cap_n I_n = \{0\}$.

Let $\mathcal{L} = M \oplus^{\mathrm{id}} H$ (see (3.10)). For each n, $J_n := I_n \oplus^{\mathrm{id}} H$ is a closed Lie ideal of finite codimension and $K := \{0\} \oplus^{\mathrm{id}} H = \cap_n J_n$ is the largest commutative closed Lie ideal of $\mathcal{L}$. Then $D^n(\mathcal{L}) \subseteq J_n$. Hence $D^{\infty}(\mathcal{L}) = \cap_n D^n(\mathcal{L}) = K$, so that $\mathcal{D}(\mathcal{L}) = D(K) = \{0\}$. Thus $\mathcal{L}$ is $\mathcal{D}$-semisimple and, hence, $\mathcal{F}$-semisimple. Apart from K, only $K_n := \{0\} \oplus^{\mathrm{id}} H_n$ for $n \in N \cup \{0\}$ are the other commutative closed Lie ideals of $\mathcal{L}$. Then $K \in \mathrm{Ess}_l(\mathcal{A}_{\mathcal{L}})$, as $\dim K/K_n = \infty$ for all n. As $\mathcal{L}/K \approx M$ is $\mathcal{F}$-semisimple, K is a $\mathcal{F}$-primitive ideal of $\mathcal{L}$. Thus $K \in \mathcal{A}_{\mathcal{L}}^{\mathrm{Prim}} \cap \mathrm{Ess}_l(\mathcal{A}_{\mathcal{L}})$. By Corollary 8.16, $\mathcal{L}$ is not strongly $\mathcal{F}$-semisimple.

The algebra $\mathcal{L}$ in the next example is $\mathcal{D}$-radical and strongly $\mathcal{F}$-semisimple.

Example 8.18. Modify the nest G in the example above as follows. Let $G = H \cup \{H_{2n}\}_{n=0}^{\infty}$. Let P_n be the orthogonal projections on H_{2n} and $Q_n = P_n - P_{n-1}$. Let $\mathcal{L}$ be the Lie algebra of all compact operators T preserving $G : TP_n = P_nTP_n$, for all n, and such that $\mathrm{Tr}(Q_nTQ_n) = 0$, for all n. Let us check that $\overline{[\mathcal{L},\mathcal{L}]} = \mathcal{L}$ whence $\mathcal{D}(\mathcal{L}) = \mathcal{L}$, so that $\mathcal{L}$ is $\mathcal{D}$-radical.

For each n, set $\mathcal{L}_n = \{T \in \mathcal{L} : T = P_nTP_n\}$. For all $T \in \mathcal{L}$, we have $TP_n \in \mathcal{L}_n$ and $TP_n \to T$. Hence $\cup_n \mathcal{L}_n$ is norm dense in $\mathcal{L}$ and it suffices to show that $[\mathcal{L}_n, \mathcal{L}_n] = \mathcal{L}_n$ for all n. Each $T \in \mathcal{L}_n$ can be realized as an upper triangular block-matrix $T = (T_{ij})$ with entries $T_{ij} = Q_iTQ_j$ in $M_2(\mathbb{C})$ whose diagonal entries T_{ii} belong to $sl(2,\mathbb{C})$ and $T_{ij} = 0$ if $i > n$, or $j > n$.

For $k \leq m \leq n$, the subspace $\mathcal{L}_n^{km} = \{T = (T_{ij}) \in \mathcal{L}_n : T_{ij} = 0 \text{ if } (i,j) \neq (k,m)\}$ of $\mathcal{L}_n$ is isomorphic to $M_2(\mathbb{C})$ if $k \neq m$, the Lie algebra $\mathcal{L}_n^{kk}$ to $sl(2,\mathbb{C})$ and $\mathcal{L}_n$ is the direct sum of all $\mathcal{L}_n^{km}$. As $[sl(2,\mathbb{C}), sl(2,\mathbb{C})] = sl(2,\mathbb{C})$ and $sl(2,\mathbb{C})M_2(\mathbb{C}) = M_2(\mathbb{C})$, we have $[\mathcal{L}_n^{kk}, \mathcal{L}_n^{kk}] = \mathcal{L}_n^{kk}$ and $[\mathcal{L}_n^{kk}, \mathcal{L}_n^{km}] = \mathcal{L}_n^{kk}\mathcal{L}_n^{km} = \mathcal{L}_n^{km}$. Thus $[\mathcal{L}_n, \mathcal{L}_n] = \mathcal{L}_n$, so that $\mathcal{L}$ is $\mathcal{D}$-radical.

Setting $I_n = \{T \in \mathcal{L} : T|_{H_{2n}} = 0\}$, we see that all I_n are closed ideals of finite codimension in $\mathcal{L}$, $I_{n+1} \subseteq I_n$ and $\cap_{n=1}^{\infty} I_n = \{0\}$, so that $\mathcal{L}$ is strongly $\mathcal{F}$-semisimple.

A closed Lie ideal I of $\mathcal{L} \in \mathfrak{L}$ is called *strongly $\mathcal{F}$-primitive* (*strongly Frattini-primitive*) if $\mathcal{L}/I$ is strongly $\mathcal{F}$-semisimple. Denote by $\mathrm{Prim}_{\mathcal{F}}^{s}(\mathcal{L})$ the set of all strongly $\mathcal{F}$-primitive ideals of $\mathcal{L}$. Then $\mathrm{Prim}_{\mathcal{F}}^{s}(\mathcal{L}) \subseteq \mathrm{Prim}_{\mathcal{F}}(\mathcal{L})$, for each $\mathcal{L} \in \mathfrak{L}$. Set

$$\mathcal{F}_s(\mathcal{L}) = \mathfrak{p}\left(\mathrm{Prim}_{\mathcal{F}}^{s}(\mathcal{L})\right) = \cap_{J \in \mathrm{Prim}_{\mathcal{F}}^{s}(\mathcal{L})} J. \tag{8.1}$$

Then

$$\mathcal{F}(\mathcal{L}) \subseteq \mathcal{F}_s(\mathcal{L}), \text{ so that } \mathcal{F} \leq \mathcal{F}_s. \tag{8.2}$$

Clearly $\mathcal{F}_s(\mathcal{L}) = \{0\}$ if and only if $\mathcal{L}$ is strongly $\mathcal{F}$-semisimple.

The following statement is similar to Lemma 8.7.

Lemma 8.19. *Let $\mathcal{L}$ be a Banach Lie algebra.*

(i) *A closed Lie ideal I of $\mathcal{L}$ is strongly $\mathcal{F}$-primitive if and only if there is a complete, lower finite-gap chain of closed Lie ideals between I and $\mathcal{L}$.*

(ii) *Let I, J be closed Lie ideal of $\mathcal{L}$, $J \subseteq I$ and $\dim(I/J) < \infty$. If I is strongly $\mathcal{F}$-primitive then J is strongly $\mathcal{F}$-primitive.*

(iii) *Each complete, lower finite-gap chain C of closed Lie ideals of $\mathcal{L}$ with $\mathfrak{s}(C) = \mathcal{L}$ consists of strongly $\mathcal{F}$-primitive ideals of $\mathcal{L}$.*

(iv) *The set $\mathrm{Prim}^s_{\mathcal{F}}(\mathcal{L})$ is $\mathfrak{p}$-complete, lower finite-gap family, $\mathfrak{s}(\mathrm{Prim}^s_{\mathcal{F}}(\mathcal{L})) = \mathcal{L}$ and $\mathcal{F}_s(\mathcal{L})$ is the smallest strongly $\mathcal{F}$-primitive Lie ideal of $\mathcal{L}$.*

(v) *Let $\mathcal{M}$ be a closed Lie subalgebra of $\mathcal{L}$. If $I \in \mathrm{Prim}^s_{\mathcal{F}}(\mathcal{L})$ then $I \cap \mathcal{M} \in \mathrm{Prim}^s_{\mathcal{F}}(\mathcal{M})$.*

(vi) *If $\mathcal{L}$ is a commutative Banach Lie algebra then $\mathcal{F}_s(\mathcal{L}) = \{0\}$.*

Proof. Parts (i)-(iii), (v) can be proved in the same way as parts (i)-(iii), (v) in Lemma 8.7.

(iv) As $\mathcal{L} \in \mathrm{Prim}^s_{\mathcal{F}}(\mathcal{L})$, we have $\mathfrak{s}(\mathrm{Prim}^s_{\mathcal{F}}(\mathcal{L})) = \mathcal{L}$. Let $G = \{I_\lambda\}_{\lambda\in\Lambda}$ be a subfamily in $\mathrm{Prim}^s_{\mathcal{F}}(\mathcal{L})$. By (i), for each I_λ, there is a complete, lower finite-gap chain C_λ of closed Lie ideals of $\mathcal{L}$ between I_λ and $\mathcal{L}$. By Proposition 6.8, $X_G := (\cup_\lambda C_\lambda)^{\mathfrak{p}}$ is a lower finite-gap family of closed Lie ideals of $\mathcal{L}$. By Lemma 6.5, X_G has a complete, lower finite-gap chain C of subspaces (i.e., closed Lie ideals of $\mathcal{L}$) between $\mathfrak{p}(X_G)$ and $\mathcal{L}$. By (i), $\mathfrak{p}(X_G) \in \mathrm{Prim}^s_{\mathcal{F}}(\mathcal{L})$. Also

$$\mathfrak{p}(X_G) = \mathfrak{p}\left((\cup_\lambda C_\lambda)^{\mathfrak{p}}\right) = \cap_\lambda \mathfrak{p}(C_\lambda) = \cap_\lambda I_\lambda = \mathfrak{p}(G).$$

Thus $\mathfrak{p}(G) \in \mathrm{Prim}^s_{\mathcal{F}}(\mathcal{L})$. Therefore $\mathrm{Prim}^s_{\mathcal{F}}(\mathcal{L})$ is $\mathfrak{p}$-complete.

Take $G = \mathrm{Prim}^s_{\mathcal{F}}(\mathcal{L})$ and let $I \in \mathrm{Prim}^s_{\mathcal{F}}(\mathcal{L})$. By the above argument, X_G is a lower finite-gap family and $\mathrm{Prim}^s_{\mathcal{F}}(\mathcal{L}) \subseteq X_G$. Then there is $J \in X_G$ such that $J \subsetneqq I$ and $\dim(I/J) < \infty$. By (ii), $J \in \mathrm{Prim}^s_{\mathcal{F}}(\mathcal{L})$. Hence $\mathrm{Prim}^s_{\mathcal{F}}(\mathcal{L})$ is a lower finite-gap family.

(vi) If $\mathcal{L}$ is commutative then, by (ii), each subspace of $\mathcal{L}$ of finite codimension is a strongly $\mathcal{F}$-primitive Lie ideal of $\mathcal{L}$. Hence, by (8.1), $\mathcal{F}_s(\mathcal{L}) = \{0\}$. $\square$

We will construct now some new examples of strongly $\mathcal{F}$-semisimple Lie algebras as the normed direct products and the c_0-direct products of strongly $\mathcal{F}$-semisimple Lie algebras. Let $\{\mathcal{L}_\lambda\}_{\lambda\in\Lambda}$ be a family of Banach Lie algebras with a bounded set of multiplication constants, let $\mathcal{L} = \oplus_\Lambda \mathcal{L}_\lambda$ and $\widehat{\mathcal{L}} = \widehat{\oplus}_\Lambda \mathcal{L}_\lambda$ (see (3.11)). For $a = (a_\lambda)_{\lambda\in\Lambda} \in \mathcal{L}$, let $\psi_\mu(a) = a_\mu$, so ψ_μ is a homomorphism from $\mathcal{L}$ to $\mathcal{L}_\mu$.

Proposition 8.20. (i) *If all $\mathcal{L}_\lambda$ are strongly $\mathcal{F}$-semisimple then $\mathcal{L}$ and $\widehat{\mathcal{L}}$ are strongly $\mathcal{F}$-semisimple.*

(ii) *If all $\mathcal{L}_\lambda$ are finite-dimensional and semisimple then*

a) $\mathcal{L}$ has a maximal lower finite-gap chain of characteristic Lie ideals from $\{0\}$ to $\mathcal{L}$;

b) $\widehat{\mathcal{L}}$ also has such a chain and is $\mathcal{D}$-radical.

Proof. For each $\mu \in \Lambda$, set $\mathcal{N}_\mu = \psi_\mu^{-1}(0)$. Then $\mathcal{L}/\mathcal{N}_\mu$ is strongly $\mathcal{F}$-semisimple, as it is isomorphic to $\mathcal{L}_\mu$. Hence $\mathcal{N}_\mu$ is a strongly $\mathcal{F}$-primitive Lie ideal. Therefore, by (8.1), $\mathcal{F}_s(\mathcal{L}) \subseteq \cap_{\mu\in\Lambda}\mathcal{N}_\mu = \{0\}$. Part (i) is proved.

If each $\mathcal{L}_\lambda$ is semisimple finite-dimensional, then $\mathcal{L}$ has no non-zero commutative Lie ideals. Hence the set $\mathcal{A}^{\mathrm{ch}}_{\mathcal{L}} = \mathcal{A}_{\mathcal{L}} = \{\{0\}\}$ is a lower finite-gap family. By Corollary 8.15, $\mathcal{L}$ has the required chain. The existence of this type of chains in $\widehat{\mathcal{L}}$ can be proved similarly. As $\mathcal{D}(\mathcal{L}_\lambda) = \mathcal{L}_\lambda$, for each λ, we have from Proposition 3.13 that $\mathcal{D}(\widehat{\mathcal{L}}) = \widehat{\oplus}_\Lambda \mathcal{D}(\mathcal{L}_\lambda) = \widehat{\mathcal{L}}$. $\square$

Let G be the set of all closed Lie ideals of $\mathcal{L}$. It is $\mathfrak{p}$-complete. Comparing (6.1), Theorem 6.11 and Lemma 8.7, we have that $G_{\mathrm{f}} = \mathrm{Prim}^s_{\mathcal{F}}(\mathcal{L})$ and $\Delta_G = \mathcal{F}_s(\mathcal{L})$. This and Lemma 6.5 yield

Corollary 8.21. (i) $\mathcal{F}_s(\mathcal{L}) = \mathfrak{p}(C)$ *for each maximal, lower finite-gap chain* C *of closed Lie ideals of* $\mathcal{L}$ *with* $\mathfrak{s}(C) = \mathcal{L}$.

(ii) *Each* $\mathfrak{p}$*-complete, lower finite-gap chain* C *of closed Lie ideals of* $\mathcal{L}$ *with* $\mathfrak{s}(C) = \mathcal{L}$ *extends to a maximal, lower finite-gap chain of closed Lie ideals of* $\mathcal{L}$.

Note that $\mathcal{F}_s(\mathcal{L})$ may have closed Lie ideals of finite codimension, but they are not Lie ideals of $\mathcal{L}$. Thus all lower finite-gap chains of closed Lie ideals end at $\mathcal{F}_s(\mathcal{L})$ and can not be extended further.

Corollary 8.22. *Each closed Lie subalgebra* $\mathcal{M}$ *of a strongly* $\mathcal{F}$*-semisimple algebra* $\mathcal{L}$ *is strongly* $\mathcal{F}$*-semisimple.*

Proof. As $\{0\} \in \mathrm{Prim}^s_{\mathcal{F}}(\mathcal{L})$, we have from Lemma 8.19(v) that $\{0\} \in \mathrm{Prim}^s_{\mathcal{F}}(\mathcal{M})$. Thus $\mathcal{M}$ is a strongly $\mathcal{F}$-semisimple. □

Theorem 8.23. $\mathcal{F}_s$ *is an over radical in* $\overline{\mathbf{L}}$ (*see Definition* 3.4).

Proof. Let $f\colon \mathcal{L} \longrightarrow \mathcal{M}$ be a morphism in $\overline{\mathbf{L}}$. By Lemma 8.19(i) and (iv), there is a complete, lower finite-gap chain C of strongly $\mathcal{F}$-primitive ideals of $\mathcal{M}$ between $\mathcal{F}_s(\mathcal{M})$ and $\mathcal{M}$. Then $C' := \left\{f^{-1}(I)\colon I \in C\right\}$ is a complete, lower finite-gap chain of closed Lie ideals between $f^{-1}(\mathcal{F}_s(\mathcal{M}))$ and $\mathcal{L}$. By Lemma 8.19(iii), C' consists of strongly $\mathcal{F}$-primitive ideals of $\mathcal{L}$. So $\mathcal{F}_s(\mathcal{L}) \subseteq f^{-1}(\mathcal{F}_s(\mathcal{M}))$, as $\mathcal{F}_s(\mathcal{L})$ is the smallest strongly $\mathcal{F}$-primitive ideal of $\mathcal{L}$ by Lemma 8.19(iv). Hence $f(\mathcal{F}_s(\mathcal{L})) \subseteq \mathcal{F}_s(\mathcal{M})$. This means that $\mathcal{F}_s$ is a preradical.

Let $J \lhd \mathcal{L}$. By Lemma 8.19(v), $I \cap J$ is a strongly $\mathcal{F}$-primitive ideal of J, for each $I \in \mathrm{Prim}^s_{\mathcal{F}}(\mathcal{L})$. Thus $\{I \cap J\colon I \in \mathrm{Prim}^s_{\mathcal{F}}(\mathcal{L})\} \subseteq \mathrm{Prim}^s_{\mathcal{F}}(J)$. Hence $\mathcal{F}_s$ is balanced, as

$$\mathcal{F}_s(J) = \mathfrak{p}(\mathrm{Prim}^s_{\mathcal{F}}(J)) \subseteq \mathfrak{p}(J \cap \mathrm{Prim}^s_{\mathcal{F}}(\mathcal{L})) = J \cap \mathfrak{p}(\mathrm{Prim}^s_{\mathcal{F}}(\mathcal{L})) = J \cap \mathcal{F}_s(\mathcal{L}) \subseteq \mathcal{F}_s(\mathcal{L}).$$

By Lemma 8.19(iv), $\mathcal{F}_s(\mathcal{L}) \in \mathrm{Prim}^s_{\mathcal{F}}(\mathcal{L})$. Hence $\mathcal{L}/\mathcal{F}_s(\mathcal{L})$ is strongly $\mathcal{F}$-semisimple. Thus $\{0\} \in \mathrm{Prim}^s_{\mathcal{F}}((\mathcal{L}/\mathcal{F}_s(\mathcal{L}))$, so that $\mathcal{F}_s(\mathcal{L}/\mathcal{F}_s(\mathcal{L})) = \{0\}$. Therefore $\mathcal{F}_s$ is an over radical. □

Consider a Banach space X as a commutative Lie algebra. Let L be a Banach Lie algebra and φ be a bounded Lie homomorphism from L into $B(X) = \mathfrak{D}(X)$. Let $\mathcal{L} = L \oplus^{\varphi} X$ (see (3.9)) be the semidirect product. Set $M = \varphi(L)$. The set Lat M of all closed subspaces of X invariant for all operators in M is $\mathfrak{p}$-complete. It follows from Corollary 6.12 that there is a subspace $\Delta_M \in$ Lat M such that $\mathfrak{p}(C) = \Delta_M$ for each maximal, lower finite-gap chain C of invariant subspaces of M with $\mathfrak{s}(C) = X$; and Δ_M has no invariant subspaces of finite codimension.

Proposition 8.24. (i) *If* L *is strongly* $\mathcal{F}$*-semisimple then* $\mathcal{F}_s(\mathcal{L}) = \{0\} \oplus^{\mathrm{id}} \Delta_M$.

(ii) *If* $\Delta_M \neq \{0\}$ (*e.g.* M *has no non-trivial invariant subspaces in* X), *then*

$$\mathcal{F}(\mathcal{L}) = \mathcal{F}_s(\mathcal{F}_s(\mathcal{L})) = \{0\} \neq \mathcal{F}_s(\mathcal{L}).$$

Proof. (i) By (3.9), any Lie ideal of $\mathcal{L}$ contained in $\{0\} \oplus^{\varphi} X$ has form $J_Z = \{0\} \oplus^{\varphi} Z$, where $Z \subseteq X$ is invariant for M, i.e., $Z \in$ Lat M. By Proposition 3.11(i), $\mathcal{F}_s(\mathcal{L}) \subseteq \mathcal{F}_s(L) \oplus^{\varphi} X = \{0\} \oplus^{\varphi} X$. Hence, since $\mathcal{F}_s(\mathcal{L})$ is a Lie ideal of $\mathcal{L}$, $\mathcal{F}_s(\mathcal{L}) = J_Y = \{0\} \oplus^{\varphi} Y$ where $Y \in$ Lat M.

As $\mathcal{F}_s(L) = \{0\}$, it follows from Corollary 8.21 that there is a maximal, lower finite-gap chain $C_M = \{I_\lambda\}$ of closed Lie ideals of L between L and $\{0\}$. Then $\widetilde{C}_M = \{I_\lambda \oplus^{\varphi} X\}$ is a maximal, lower finite-gap chain of closed Lie ideals of $\mathcal{L}$ between $\mathcal{L}$ and $\{0\} \oplus^{\varphi} X$.

Let $C_\Delta = \{L_\mu\}$ be some maximal, lower finite-gap chain of invariant subspaces of M with $\mathfrak{s}(C_\Delta) = X$. By Corollary 6.12, $\mathfrak{p}(C_\Delta) = \Delta_M$. Hence $\widetilde{C}_\Delta = \{\{0\} \oplus^{\varphi} L_\mu\}$ is a maximal, lower finite-gap chain of Lie ideals of $\mathcal{L}$ in $\{0\} \oplus^{\varphi} X$ and $\mathfrak{p}(\widetilde{C}_\Delta) = \{0\} \oplus^{\varphi} \Delta_M$. Therefore $C = \widetilde{C}_M \cup \widetilde{C}_\Delta$ is a maximal, lower finite-gap chain of Lie ideals of $\mathcal{L}$, $\mathfrak{p}(C) = \{0\} \oplus^{\varphi} \Delta_M$ and $\mathfrak{s}(C) = \mathcal{L}$. By Corollary 8.21, $\mathcal{F}_s(\mathcal{L}) = \mathfrak{p}(C) = \{0\} \oplus^{\varphi} \Delta_M = \{0\} \oplus^{\mathrm{id}} \Delta_M$.

(ii) By (i) and Lemma 8.19(vi), $\mathcal{F}_s(\mathcal{F}_s(\mathcal{L})) = \mathcal{F}_s\left(\{0\} \oplus^{\mathrm{id}} \Delta_M\right) = \{0\} \neq \mathcal{F}_s(\mathcal{L})$. By Example 7.14(iii), $\mathcal{F}(\mathcal{L}) = \{0\}$. □

It follows from Proposition 8.24 that $\mathcal{F}_s$ is not a radical and $\mathcal{F}_s(\mathcal{F}_s(L \oplus^{\varphi} X)) = \mathcal{F}(L \oplus^{\varphi} X)$. As the following theorem shows, the equality $\mathcal{F}_s(\mathcal{F}_s(\mathcal{L})) = \mathcal{F}(\mathcal{L})$ holds for all $\mathcal{L} \in \mathfrak{L}$.

Theorem 8.25. *For each algebra $\mathcal{L} \in \mathfrak{L}$, the quotient Lie algebra $\mathcal{F}_s(\mathcal{L})/\mathcal{F}(\mathcal{L})$ is commutative,*

$$\mathcal{F}_s(\mathcal{L}/\mathcal{F}(\mathcal{L})) = \mathcal{F}_s(\mathcal{L})/\mathcal{F}(\mathcal{L}) = \mathfrak{s}\left(\mathrm{Ess}_l\left(\mathcal{A}_{\mathcal{L}/\mathcal{F}(\mathcal{L})}\right)\right) \quad \textit{and} \quad \mathcal{F}_s(\mathcal{F}_s(\mathcal{L})) = \mathcal{F}(\mathcal{L}). \tag{8.3}$$

Proof. Firstly assume that $\mathcal{L}$ is $\mathcal{F}$-semisimple. Then $\mathcal{F}(\mathcal{L}) = \{0\}$ and we have to show that

$$\mathcal{F}_s(\mathcal{L}) \text{ is commutative, } \mathcal{F}_s(\mathcal{L}) = \mathfrak{s}(\mathrm{Ess}_l(\mathcal{A}_{\mathcal{L}})) \text{ and } \mathcal{F}_s(\mathcal{F}_s(\mathcal{L})) = \{0\}. \tag{8.4}$$

Let $I \in \mathrm{Ess}_l(\mathcal{A}_{\mathcal{L}})$. Then $\mathcal{L}$ has no Lie ideals contained in I that have finite, non-zero codimension in I. By Lemma 8.19(iv), $\mathrm{Prim}^s_{\mathcal{F}}(\mathcal{L})$ is a lower finite-gap family of Lie ideals of $\mathcal{L}$. Hence, by Corollary 6.7, $I \cap \mathrm{Prim}^s_{\mathcal{F}}(\mathcal{L}) := \{I \cap J\colon\ J \in \mathrm{Prim}^s_{\mathcal{F}}(\mathcal{L})\}$ is also a lower finite-gap family of Lie ideals of $\mathcal{L}$ and I belongs to it, as $\mathcal{L} \in \mathrm{Prim}^s_{\mathcal{F}}(\mathcal{L})$. Thus $I \cap \mathrm{Prim}^s_{\mathcal{F}}(\mathcal{L}) = \{I\}$, so that I lies in each J in $\mathrm{Prim}^s_{\mathcal{F}}(\mathcal{L})$. Hence $I \subseteq \mathfrak{p}(\mathrm{Prim}^s_{\mathcal{F}}(\mathcal{L})) = \mathcal{F}_s(\mathcal{L})$. As I is arbitrary, $\mathfrak{s}(\mathrm{Ess}_l(\mathcal{A}_{\mathcal{L}})) \subseteq \mathcal{F}_s(\mathcal{L})$. Set $K = \mathfrak{s}(\mathrm{Ess}_l(\mathcal{A}_{\mathcal{L}}))$.

Let a Lie ideal I contain K. If I contains a Lie ideal J of non-zero, finite codimension in I, then $K \subseteq J$. Indeed, if $L \in \mathrm{Ess}_l(\mathcal{A}_{\mathcal{L}})$ then $L \subseteq J$; otherwise, by Lemma 6.1, $L \cap J$ has non-zero, finite codimension in L which contradicts the fact that $L \in \mathrm{Ess}_l(\mathcal{A}_{\mathcal{L}})$. Hence $K \subseteq J$.

Assume that $K \neq I$. If $I \in \mathcal{A}_{\mathcal{L}}$ then I contains a Lie ideal $J \in \mathcal{A}_{\mathcal{L}}$ of non-zero, finite codimension in I. By the above, $K \subseteq J$. Let I be non-commutative, i.e., $I \in \mathrm{Lid}(\mathcal{L}) \diagdown \mathcal{A}_{\mathcal{L}}$. By Proposition 8.11(i), $\mathrm{Lid}(\mathcal{L}) \diagdown \mathcal{A}_{\mathcal{L}}$ is a lower finite-gap family modulo $\mathcal{A}_{\mathcal{L}}$. Hence I contains a Lie ideal J that has non-zero, finite codimension in I. By the above, $K \subseteq J$.

Thus the set $\{I\colon\ I \lhd \mathcal{L} \text{ and } K \subseteq I\}$ is a $\mathfrak{p}$-complete, lower finite-gap family. By Lemma 6.5, there is a complete, lower finite-gap chain of closed Lie ideals between K and $\mathcal{L}$. Hence K is strongly $\mathcal{F}$-primitive by Lemma 8.19(i). Therefore $\mathcal{F}_s(\mathcal{L}) \subseteq K$. Thus we have finally that $\mathcal{F}_s(\mathcal{L}) = K = \mathfrak{s}(\mathrm{Ess}_l(\mathcal{A}_{\mathcal{L}}))$.

By Lemma 8.19(iv), $\mathcal{F}_s(\mathcal{L})$ is the smallest strongly $\mathcal{F}$-primitive ideal of $\mathcal{L}$. Hence, by Lemma 8.19(ii),

$$\mathcal{F}_s(\mathcal{L}) \text{ contains no closed Lie ideals of } \mathcal{L} \text{ of non-zero finite codimension.} \tag{8.5}$$

Let $\dim \mathcal{F}_s(\mathcal{L}) < \infty$. As $\{0\}$ is a Lie ideal of finite codimension in $\mathcal{F}_s(\mathcal{L})$, we have from (8.5) that $\mathcal{F}_s(\mathcal{L}) = \{0\}$ and (8.4) holds.

Let $\dim \mathcal{F}_s(\mathcal{L}) = \infty$. If $\mathcal{F}_s(\mathcal{L})$ is not commutative, it has a closed Lie subalgebra of non-zero, finite codimension by Theorem 8.6(v). Hence, by Corollary 2.6, $\mathcal{F}_s(\mathcal{L})$ contains a closed Lie ideal of $\mathcal{L}$ of non-zero, finite codimension. This contradicts (8.5) and shows that $\mathcal{F}_s(\mathcal{L})$ is commutative. By Lemma 8.19(vi), $\mathcal{F}_s(\mathcal{F}_s(\mathcal{L})) = \{0\}$ and (8.4) is proved.

Suppose now that $\mathcal{L}$ is not $\mathcal{F}$-semisimple. Let $q\colon\ \mathcal{L} \to \mathcal{M} := \mathcal{L}/\mathcal{F}(\mathcal{L})$. If $I \in \mathrm{Prim}^s_{\mathcal{F}}(\mathcal{M})$ then $\mathcal{M}/I$ is strongly $\mathcal{F}$-semisimple. As $\mathcal{M}/I \approx \mathcal{L}/q^{-1}(I)$, we have $q^{-1}(I) \in \mathrm{Prim}^s_{\mathcal{F}}(\mathcal{L})$.

Conversely, let $J \in \mathrm{Prim}^s_{\mathcal{F}}(\mathcal{L})$. As $\mathrm{Prim}^s_{\mathcal{F}}(\mathcal{L}) \subseteq \mathrm{Prim}_{\mathcal{F}}(\mathcal{L})$ and (see Lemma 8.7(iv)) $\mathcal{F}(\mathcal{L}) = \mathfrak{p}(\mathrm{Prim}_{\mathcal{F}}(\mathcal{L}))$, we have $\mathcal{F}(\mathcal{L}) \subseteq J$. As $\mathcal{M}/q(J) \approx \mathcal{L}/J$, we have $q(J) \in \mathrm{Prim}^s_{\mathcal{F}}(\mathcal{M})$. Thus, by (8.1),

$$\mathcal{F}_s(\mathcal{M}) = \bigcap_{I \in \mathrm{Prim}^s_{\mathcal{F}}(\mathcal{M})} I = \bigcap_{J \in \mathrm{Prim}^s_{\mathcal{F}}(\mathcal{L})} q(J) = q(\mathcal{F}_s(\mathcal{L})) = \mathcal{F}_s(\mathcal{L})/\mathcal{F}(\mathcal{L}).$$

The Lie algebra $\mathcal{M}$ is $\mathcal{F}$-semisimple, as $\mathcal{F}$ is a radical. Hence it follows from (8.4) that the Lie ideal $\mathcal{F}_s(\mathcal{M}) = \mathcal{F}_s(\mathcal{L})/\mathcal{F}(\mathcal{L})$ is commutative, (8.3) holds and

$$\mathcal{F}_s(\mathcal{F}_s(\mathcal{L})/\mathcal{F}(\mathcal{L})) = \mathcal{F}_s(\mathcal{F}_s(\mathcal{M})) = \{0\}. \tag{8.6}$$

Set $I = \mathcal{F}(\mathcal{L})$ and $L = \mathcal{F}_s(\mathcal{L})$. Then (8.6) turns into $\mathcal{F}_s(L/I) = \{0\}$. By (8.2), $\mathcal{F} \leq \mathcal{F}_s$, so that $I \lhd L$. As $\mathcal{F}$ is a radical, $I = \mathcal{F}(\mathcal{L}) = \mathcal{F}(\mathcal{F}(\mathcal{L})) = \mathcal{F}(I)$. Hence, by Proposition 3.7(v), $\mathcal{F}(\mathcal{L}) = I = \mathcal{F}_s(L) = \mathcal{F}_s(\mathcal{F}_s(\mathcal{L}))$. The proof is complete. $\square$

Let $\mathcal{L}$ be an $\mathcal{F}$-semisimple Banach algebra and $\mathcal{F}_s(\mathcal{L}) \neq \{0\}$. By Theorem 8.25, $\mathcal{F}_s(\mathcal{L})$ is commutative. Let $L = \mathcal{L}/\mathcal{F}_s(\mathcal{L})$ and $q\colon\ \mathcal{L} \longrightarrow L$ be the quotient map. As $\mathcal{F}_s$ is an over radical, $\mathcal{F}_s(L) = \{0\}$. Thus each $\mathcal{F}$-semisimple Banach algebra $\mathcal{L}$ is an *extension* of a commutative Banach Lie algebra $\mathcal{F}_s(\mathcal{L})$ by an $\mathcal{F}_s$-semisimple algebra.

Associate with $\mathcal{L}$ the semidirect product $\mathcal{N} = L \oplus^{\varphi} \mathcal{F}_s(\mathcal{L})$ (see (3.9)) in the following way. As $\mathcal{F}_s(\mathcal{L})$ is commutative, the map φ from L into the Lie algebra $\mathfrak{D}(\mathcal{F}_s(\mathcal{L})) = B(\mathcal{F}_s(\mathcal{L}))$ of all bounded derivations on $\mathcal{F}_s(\mathcal{L})$ defined by $\varphi(q(a)) = \delta_a|_{\mathcal{F}_s(\mathcal{L})}$, where $\delta_a(x) = [a, x]$ for

$x \in \mathcal{F}_s(\mathcal{L})$, is a correctly defined Lie homomorphism. Hence $\mathcal{N}$ is well defined. If $\mathcal{L}$ has a closed Lie subalgebra topologically isomorphic to L, then $\mathcal{L}$ is topologically isomorphic to $\mathcal{N}$.

By Proposition 3.11(ii) 3), $\mathcal{N}$ is $\mathcal{F}$-semisimple. Moreover, $\mathcal{F}_s(\mathcal{N}) = \{0\} \oplus^{\varphi} \mathcal{F}_s(\mathcal{L})$. Indeed, let $M = \varphi(L)$. The lattice Lat M of invariant subspaces in $\mathcal{F}_s(\mathcal{L})$ for M coincides with the lattice of Lie ideals of $\mathcal{L}$ in $\mathcal{F}_s(\mathcal{L})$. By Corollary 8.21, $\mathcal{F}_s(\mathcal{L})$ contains no Lie ideals of $\mathcal{L}$ of non-zero, finite codimension. Hence Lat M has no subspaces of finite codimension. Thus (see Proposition 8.24) the subspace $\Delta_M = \mathcal{F}_s(\mathcal{L})$ and $\mathcal{F}_s(\mathcal{N}) = \{0\} \oplus^{\varphi} \mathcal{F}_s(\mathcal{L})$.

9. The structure of Frattini-free Banach Lie algebras

In this section we study a special subclass of Frattini-semisimple Lie algebras that consists of Frattini-free Lie algebras. A Banach Lie algebra $\mathcal{L}$ is called *Frattini-free* if it has sufficiently many maximal closed Lie subalgebras of finite codimension, that is,

$$P_{\mathfrak{S}^{\max}}(\mathcal{L}) = \cap_{L \in \mathfrak{S}^{\max}_{\mathcal{L}}} L = \{0\}, \text{ or } \mathcal{L} \in \mathbf{Sem}(P_{\mathfrak{S}^{\max}}).$$

We also will use the term *Jacobson-free* for Banach Lie algebras in $\mathbf{Sem}(P_{\mathfrak{J}^{\max}})$.

Marshall in [M, p. 417] proved that all *simple* finite-dimensional Lie algebras are Frattini-free. In Theorem 9.9 we give a full description of Frattini-free Lie algebras: they are isomorphic to subdirect products of the normed direct products of finite-dimensional subsimple Lie algebras.

9.1. **Subsimple algebras and submaximal ideals.** Frattini-free algebras need not have sufficiently many maximal Lie ideals (see Example 7.20(iii)). Instead they have sufficiently many *submaximal* ideals (see Theorem 9.9 below).

Definition 9.1. (i) *We call a finite-dimensional Lie algebra $\mathcal{L}$* subsimple *if either* $\dim \mathcal{L} = 1$ *or it has a proper maximal Lie subalgebra that contains no non-zero Lie ideals of $\mathcal{L}$.*

(ii) *We call a closed Lie ideal J of $\mathcal{L} \in \mathfrak{L}$* submaximal, *if $\mathcal{L}/J$ is a subsimple Lie algebra.*

Lemma 9.2. *Each subsimple Lie algebra $\mathcal{L}$ is Frattini-free. Its centre $Z = \{0\}$ if* $\dim \mathcal{L} \geq 2$.

Proof. Let $\dim \mathcal{L} \geq 2$ and let a maximal Lie subalgebra M of $\mathcal{L}$ contain no non-zero Lie ideals of $\mathcal{L}$. As $P_{\mathfrak{S}^{\max}}(\mathcal{L}) = \cap_{L \in \mathfrak{S}^{\max}_{\mathcal{L}}} L$ is a Lie ideal of $\mathcal{L}$ contained in M, we have $P_{\mathfrak{S}^{\max}}(\mathcal{L}) = \{0\}$.

If $Z \neq \{0\}$ then $M \cap Z = \{0\}$, as $M \cap Z$ is a Lie ideal of $\mathcal{L}$. Hence $Z \dot{+} M = \mathcal{L}$ and $\dim(Z) = 1$ by maximality of M. Thus M itself is a Lie ideal of $\mathcal{L}$ – a contradiction. Hence $Z = \{0\}$. $\square$

It follows from the definition that simple Lie algebras are subsimple. To clarify the structure of subsimple Lie algebras, consider the following classes of finite-dimensional Lie algebras:

(**I**) *Class* (**I**) *consists of Lie algebras $\mathcal{L} = N \oplus N$, where N is a simple Lie algebra.*

(**II**) *Class* (**II**) *consists of Lie algebras $\mathcal{L} = N \oplus^{\mathrm{id}} X$, where N is a Lie algebra of operators on a finite-dimensional space X which has no non-trivial invariant subspaces.*

Lemma 9.3. *All Lie algebras in classes* (**I**) *and* (**II**) *are subsimple.*

Proof. Let $\mathcal{L} = N \oplus N$. Clearly, $N \oplus \{0\}$ and $\{0\} \oplus N$ are the only non-trivial Lie ideals of $\mathcal{L}$. Hence the Lie subalgebra $M = \{a \oplus a\colon a \in N\}$ does not contain non-zero Lie ideals. To see that it is maximal note that, for each $b = a_1 \oplus a_2 \notin M$, the subalgebra M' generated by $\{b\} \cup M$ contains all elements of the form $0 \oplus [a, a_2 - a_1]$ where $a \in N$. Since N is simple, M' contains $\{0\} \oplus N$, whence $M' = \mathcal{L}$. Thus $\mathcal{L}$ is subsimple.

Let now $\mathcal{L} = N \oplus^{\mathrm{id}} X$. The Lie ideal $\{0\} \oplus^{\mathrm{id}} X$ is contained in every Lie ideal of $\mathcal{L}$, so that the Lie subalgebra $M = N \oplus^{\mathrm{id}} \{0\}$ contains no non-zero Lie ideals of $\mathcal{L}$. If M' is a Lie subalgebra that contains M then the subspace $Y = \{x \in X\colon 0 \oplus x \in M'\}$ is invariant for N. Hence either $Y = \{0\}$ and $M' = M$, or $Y = X$ and $M' = \mathcal{L}$. Thus M is maximal, whence $\mathcal{L}$ is subsimple. $\square$

Now we will show that our list of subsimple Lie algebras is exhausting.

Theorem 9.4. *Let $\dim \mathcal{L} \geq 2$. Then $\mathcal{L}$ is subsimple if and only if it is either simple, or isomorphic to a Lie algebra from classes* (**I**) *or* (**II**). *More precisely, if $\mathcal{L}$ is semisimple and not simple, it is isomorphic to a Lie algebra in the class* (**I**); *if $\mathcal{L}$ is neither simple, nor semisimple, it is isomorphic to a Lie algebra in the class* (**II**).

Proof. We saw above that simple Lie algebras and Lie algebras from the classes **(I)** and **(II)** are subsimple.

Conversely, let $\mathcal{L}$ be subsimple and let M be a maximal Lie subalgebra that does not contain non-zero Lie ideals of $\mathcal{L}$. Assume firstly that $\mathcal{L}$ is not semisimple. Then $\mathcal{L}$ has a proper non-zero minimal commutative Lie ideal X. Since M is maximal, $M + X = \mathcal{L}$. Let $I = \{a \in \mathcal{L}\colon [a, X] = 0\}$. Then I is a Lie ideal of $\mathcal{L}$. We also have that $M \cap I$ is a Lie ideal of $\mathcal{L}$, since

$$[M \cap I, \mathcal{L}] = [M \cap I, M + X] = [M \cap I, M] \subseteq M \cap I.$$

Therefore $M \cap I = \{0\}$; in particular, $M \cap X = \{0\}$, so that the sum $L = M + X$ is direct. Moreover, the equality $M \cap I = \{0\}$ shows that the map $a \longmapsto \mathrm{ad}\,(a)\,|_X$ is injective on M. Set $N = \mathrm{ad}\,(M)\,|_X$. Then $\mathcal{L}$ is isomorphic to $N \oplus^{\mathrm{id}} X$ and belongs to class **(II)**.

Assume that $\mathcal{L}$ is semisimple, but not simple. Then $\mathcal{L} = L_1 \oplus \cdots \oplus L_n$ is the direct sum of simple Lie algebras L_i and $n \geq 2$. As each L_i is a Lie ideal of $\mathcal{L}$, we have $\mathcal{L} = M + L_i$. Let P_j be the natural projection $x_1 \oplus \cdots \oplus x_n \longmapsto x_j$ from $\mathcal{L}$ onto L_j, where $x_j \in L_j$. Then $P_j\,(M) = L_j$. Let $K_j = M \cap L_j$. Then K_j is a Lie ideal of M, whence $[K_j, L_j] = [K_j, P_j\,(M)] = [K_j, M] \subseteq K_j$. Thus K_j is a Lie ideal of L_j. Hence K_j is a Lie ideal of $M + L_j = \mathcal{L}$. As M contains no non-zero Lie ideals of $\mathcal{L}$, we have $K_j = \{0\}$ for all j.

Fix i and j for $j \neq i$. Then $L_j \subseteq M + L_i$. Hence, for each $x \in L_j$, there is $y \in L_i$ such that $x + y \in M$. Combining this with the fact that $M \cap L_j = \{0\}$ for all j, we have that there is an injective Lie homomorphism φ_{ij} from L_j into L_i such that $\{x \oplus \varphi_{ij}(x)\colon x \in L_j\} \subseteq M$ (as each $\varphi_{ij}(x) \in L_i$, we have that all such $x \oplus \varphi_{ij}(x)$ lie in $L_j \oplus L_i$). Exchanging i and j, we have that φ_{ij} is a Lie isomorphism. Thus all L_j are isomorphic.

If $n \geq 3$, set $\psi = \varphi_{21}$ and $\omega = \varphi_{31}$. Then $x_\psi := x \oplus \psi(x) \in M$ and $x_\omega := x \oplus \omega(x) \in M$ for every $x \in L_1$. Therefore $x_\psi - x_\omega = \psi(x) \oplus (-\omega(x)) \in M$ for all $x \in L_1$. Hence $[x_\psi - x_\omega, y_\psi - y_\omega] = \psi([x, y]) \oplus \omega([x, y])) \in M$ for all $x, y \in L_1$. On the other hand, $[x, y]_\psi - [x, y]_\omega = \psi([x, y]) \oplus (-\omega([x, y])) \in M$. Therefore $\omega([x, y])) \in M$ for all $x, y \in L_1$. Thus $M \cap L_3 \neq \{0\}$. This contradiction shows that $n \leq 2$ and $\mathcal{L}$ is isomorphic to an algebra from class **(I)**. □

Corollary 9.5. *If a subsimple Lie algebra $\mathcal{L}$ is solvable then* $\dim(\mathcal{L}) \leq 2$. *If it is nilpotent,* $\dim \mathcal{L} \leq 1$.

Proof. Let $\dim(\mathcal{L}) > 1$. As $\mathcal{L}$ is solvable, we have from Theorem 9.4 that $\mathcal{L} = N \oplus^{\mathrm{id}} X$ and the operator Lie algebra N has no non-trivial invariant subspaces in X. Since $\mathcal{L}$ is solvable, N is also solvable and, by the Lie Theorem, N always has a one-dimensional invariant subspace. Thus $\dim X = 1$. As $N \subseteq B(X)$, we have $\dim N = 1$, so that $\dim \mathcal{L} = 2$. If $\mathcal{L}$ is nilpotent then, as $\dim \mathcal{L} = 2$, it is commutative which contradicts Lemma 9.2. □

Let $\mathcal{L} \in \mathfrak{L}$ and $\mathcal{M} \in \mathfrak{S}^{\max}_{\mathcal{L}}$. If $\mathcal{M}$ is a Lie ideal, $\mathcal{L}/\mathcal{M}$ has no proper Lie subalgebras. Hence

$$\dim\,(\mathcal{L}/\mathcal{M}) = 1. \tag{9.1}$$

Denote by $\mathfrak{J}^{\mathrm{sm}}_{\mathcal{L}}$ the set of all submaximal Lie ideals of $\mathcal{L}$. Then $\mathfrak{J}^{\max}_{\mathcal{L}} \subseteq \mathfrak{J}^{\mathrm{sm}}_{\mathcal{L}} \subseteq \mathfrak{J}_{\mathcal{L}}$. The following result strengthens Theorem 2.5 — the central result of [KST2].

Proposition 9.6. (i) *Every maximal closed subalgebra of finite codimension in a Banach Lie algebra $\mathcal{L}$ contains a submaximal Lie ideal of $\mathcal{L}$.*

(ii) *Conversely, for each submaximal Lie ideal J of $\mathcal{L}$, there is $\mathcal{M} \in \mathfrak{S}^{\max}_{\mathcal{L}}$ such that J is a maximal element in the set of all closed Lie ideals of $\mathcal{L}$ contained in $\mathcal{M}$.*

Proof. (i) If $\dim(\mathcal{L}) < \infty$ then each maximal Lie ideal of $\mathcal{L}$ contained in every maximal subalgebra of $\mathcal{L}$ is submaximal.

Let $\dim(\mathcal{L}) = \infty$ and $\mathcal{M} \in \mathfrak{S}^{\max}_{\mathcal{L}}$. If $\mathcal{M}$ is a Lie ideal then, by (9.1), $\dim(\mathcal{L}/\mathcal{M}) = 1$. Thus $\mathcal{M}$ is a submaximal Lie ideal. If $\mathcal{M}$ is not a Lie ideal of $\mathcal{L}$, it follows from Theorem 2.5(i) that $\mathcal{M}$ contains a closed Lie ideal of $\mathcal{L}$ of finite codimension. Hence $\mathcal{M}$ contains a largest closed Lie ideal J of $\mathcal{L}$ of finite codimension. Then $\mathcal{L}/J$ is finite-dimensional and $\mathcal{M}/J$ is a maximal Lie subalgebra of $\mathcal{L}/J$ that contains no non-zero Lie ideals of $\mathcal{L}/J$. Thus $\mathcal{L}/J$ is subsimple, so that J is submaximal.

(ii) Let $J \in \mathfrak{J}^{\mathrm{sm}}_{\mathcal{L}}$. If $J \notin \mathfrak{S}^{\max}_{\mathcal{L}}$ then, by (9.1), $\dim \mathcal{L}/J \neq 1$ and there is a maximal proper Lie subalgebra M of $\mathcal{L}/J$ that contains no non-zero Lie ideals of $\mathcal{L}/J$. Then the preimage $\mathcal{M}$ of M in $\mathcal{L}$ belongs to $\mathfrak{S}^{\max}_{\mathcal{L}}$ and J is a maximal Lie ideal of $\mathcal{L}$ contained in $\mathcal{M}$. □

We will show now that the Lie ideal-multifunctions $\mathfrak{J}^{\mathrm{sm}}$ and $\mathfrak{S}^{\max}$ generate equal preradicals.

Proposition 9.7. $P_{\mathfrak{S}^{\max}}(\mathcal{L}) = P_{\mathfrak{J}^{\mathrm{sm}}_{\mathcal{L}}}(\mathcal{L})$ *for all Banach Lie algebras* $\mathcal{L}$, *so that* $P_{\mathfrak{S}^{\max}} = P_{\mathfrak{J}^{\mathrm{sm}}_{\mathcal{L}}}$.

Proof. By Proposition 9.6(i), $\mathfrak{J}^{\mathrm{sm}}_{\mathcal{L}} \overleftarrow{\subset} \mathfrak{S}^{\max}_{\mathcal{L}}$ for all $\mathcal{L} \in \mathfrak{L}$. Hence, by (5.2), $P_{\mathfrak{J}^{\mathrm{sm}}}(\mathcal{L}) \subseteq P_{\mathfrak{S}^{\max}}(\mathcal{L})$.

On the other hand, let $J \in \mathfrak{J}^{\mathrm{sm}}_{\mathcal{L}}$. By Proposition 9.6(ii), there exists $\mathcal{M} \in \mathfrak{S}^{\max}_{\mathcal{L}}$ such that J is maximal among closed Lie ideals of $\mathcal{L}$ contained in $\mathcal{M}$. As J has finite codimension in $\mathcal{L}$, we have from Lemma 6.1 that $J + P_{\mathfrak{S}^{\max}}(\mathcal{L})$ is closed. Since $P_{\mathfrak{S}^{\max}}(\mathcal{L}) \subseteq \mathcal{M}$, $J + P_{\mathfrak{S}^{\max}}(\mathcal{L})$ is a closed Lie ideal of $\mathcal{L}$ contained in $\mathcal{M}$. As J is a maximal such Lie ideal in $\mathcal{M}$, $P_{\mathfrak{S}^{\max}}(\mathcal{L}) \subseteq J$. Thus $P_{\mathfrak{S}^{\max}}(\mathcal{L}) \subseteq J$ for all $J \in \mathfrak{J}^{\mathrm{sm}}_{\mathcal{L}}$. Hence $P_{\mathfrak{S}^{\max}}(\mathcal{L}) \subseteq P_{\mathfrak{J}^{\mathrm{sm}}_{\mathcal{L}}}(\mathcal{L})$. □

As a consequence of the above proposition, $\mathbf{Sem}(P_{\mathfrak{S}^{\max}})$ coincides with the class of all algebras with sufficiently many submaximal ideals: the intersection of submaximal ideals equals zero.

9.2. **Subdirect products.** To describe Frattini-free Lie algebras in a more constructive way we need the following definition. Let $\{\mathcal{L}_\lambda\}_{\lambda\in\Lambda}$ be a family of Banach Lie algebras with multiplication constants t_λ satisfying $\sup\{t_\lambda\} < \infty$. Let $\mathcal{L}_\Lambda = \oplus_\Lambda \mathcal{L}_\lambda$ be their normed direct product (see Subsection 3.2). For each $\mu \in \Lambda$, denote by ψ_μ the homomorphism from $\mathcal{L}_\Lambda$ onto $\mathcal{L}_\mu$:

$$\psi_\mu(\{a_\lambda\}) = a_\mu. \tag{9.2}$$

Definition 9.8. *A Lie subalgebra* $\mathcal{M}$ *of* $\mathcal{L}_\Lambda = \oplus_\Lambda \mathcal{L}_\lambda$ *is called a subdirect product of the algebras* $\{\mathcal{L}_\lambda\}_{\lambda\in\Lambda}$ *if* $\psi_\mu(\mathcal{M}) = \mathcal{L}_\mu$ *for each* $\mu \in \Lambda$.

Theorem 9.9. *A Banach Lie algebra* $\mathcal{L}$ *belongs to* $\mathbf{Sem}(P_{\mathfrak{S}^{\max}})$ *if and only if there is a bounded isomorphism* θ *from* $\mathcal{L}$ *onto a subdirect product of some family of subsimple Lie algebras.*

It belongs to $\mathbf{Sem}(P_{\mathfrak{J}^{\max}})$ (*respectively, to* $\mathbf{Sem}(P_{\mathfrak{J}})$) *if and only if there is a bounded isomorphism from* $\mathcal{L}$ *onto a subdirect product of some family of simple or one-dimensional* (*respectively, finite-dimensional*) *Lie algebras.*

Proof. We will consider the case when $\mathcal{L} \in \mathbf{Sem}(P_{\mathfrak{S}^{\max}})$; the two remaining cases can be proved similarly.

Let θ and $\mathcal{L}_\Lambda$ exist. For each $\lambda \in \Lambda$, $\psi_\lambda \circ \theta$ is a bounded homomorphism from $\mathcal{L}$ onto $\mathcal{L}_\lambda$. Then $J_\lambda := \ker(\psi_\lambda \circ \theta)$ is a closed Lie ideal of $\mathcal{L}$ and $\mathcal{L}/J_\lambda$ is isomorphic to $\mathcal{L}_\lambda$, so that J_λ is submaximal. Also $\cap_{\lambda\in\Lambda} J_\lambda = \{0\}$. By Lemma 9.2, $P_{\mathfrak{S}^{\max}}(\mathcal{L}/J_\lambda) = \{0\}$. Therefore, by Lemma 3.6(ii), $P_{\mathfrak{S}^{\max}}(\mathcal{L}) \subseteq \cap_{\lambda\in\Lambda} J_\lambda = \{0\}$. Thus $\mathcal{L} \in \mathbf{Sem}(P_{\mathfrak{S}^{\max}})$.

Conversely, let $\mathcal{L} \in \mathbf{Sem}(P_{\mathfrak{S}^{\max}})$. By Proposition 9.7, $\cap_{J\in\mathfrak{J}^{\mathrm{sm}}_{\mathcal{L}}} J = P_{\mathfrak{J}^{\mathrm{sm}}}(\mathcal{L}) = P_{\mathfrak{S}^{\max}}(\mathcal{L}) = \{0\}$. Choose any subset Λ of $\mathfrak{J}^{\mathrm{sm}}_{\mathcal{L}}$ such that $\cap_{J\in\Lambda} J = \{0\}$. For each $J \in \Lambda$, the quotient Lie algebra $\mathcal{L}/J$ is subsimple. Let $q_J\colon \mathcal{L} \longrightarrow \mathcal{L}/J$ be the quotient map. Then, for $a, b \in \mathcal{L}$,

$$\begin{aligned} \|[q_J(a), q_J(b)]\|_{\mathcal{L}/J} &= \|q_J([a,b])\|_{\mathcal{L}/J} = \inf_{z\in J} \|[a,b]+z\|_{\mathcal{L}} \le \inf_{z,u\in J} \|[a+z, b+u]\|_{\mathcal{L}} \\ &\le t \inf_{z,u\in J} \|a+z\|_{\mathcal{L}} \|b+u\|_{\mathcal{L}} = t \|q_J(a)\|_{\mathcal{L}/J} \|q_J(b)\|_{\mathcal{L}/J}. \end{aligned}$$

Hence $t_J \le t$ for all $J \in \Lambda$. Thus $\sup\{t_J\colon J \in \Lambda\} < \infty$. Let $\mathcal{L}_\Lambda := \oplus_\Lambda(\mathcal{L}/J)$ be the normed direct product. For each $a \in \mathcal{L}$, we have $\{q_J(a)\}_{J\in\Lambda} \in \mathcal{L}_\Lambda$ and the map $\theta\colon a \longrightarrow \{q_J(a)\}_{J\in\Lambda}$ is bounded homomorphism. If $\theta(a) = 0$ then $q_J(a) = 0$ for all $J \in \Lambda$, so that $a \in \cap_{J\in\Lambda} J = \{0\}$. Thus θ is an isomorphism. As $\psi_J(\theta(a)) = q_J(a)$, we have $\psi_J(\theta(\mathcal{L})) = q_J(\mathcal{L}) = \mathcal{L}_J$. Hence the Lie algebra $\theta(\mathcal{L})$ is a semidirect product of the algebras $\{\mathcal{L}_\lambda\}_{\lambda\in\Lambda}$. □

As an illustration, consider the Lie algebra $\mathcal{L} = \mathbb{C}U \oplus^{\mathrm{id}} l^2$, where U is the unilateral shift: $Ue_n = e_{n+1}$ and $(e_n)_{n=1}^\infty$ is a basis in l^2. Then $\{0\} \oplus^{\mathrm{id}} l^2$ is a maximal Lie subalgebra of $\mathcal{L}$. Let $\mathbb{D} \subset \mathbb{C}$ be the open unit disk. For each $\lambda \in \mathbb{D}$, the vector $e_\lambda = (1, \lambda, \lambda^2, ...)$ belongs to l^2 and the subspace $E_\lambda = \mathbb{C}e_\lambda$ is invariant for the adjoint operator U^*: $U^*e_\lambda = \lambda e_\lambda$. Hence $E_\lambda^\perp$ is

invariant for U and has codimension 1 in l^2. Thus (see (3.10)) $L_\lambda = \mathbb{C}U \oplus^{\mathrm{id}} E_\lambda^\perp$ is a maximal Lie subalgebra of $\mathcal{L}$ of codimension 1. The Lie algebra $\mathcal{L}$ is Frattini-free, since

$$P_{\mathfrak{S}^{\max}}(\mathcal{L}) \subseteq (\{0\} \oplus^{\mathrm{id}} l^2) \cap (\cap_{\lambda\in\mathbb{D}} L_\lambda) = \{0\}.$$

To map $\mathcal{L}$ onto a subdirect product of subsimple Lie algebras, consider two-dimensional Lie algebras $\mathcal{L}_\lambda = \mathbb{C}U_\lambda \oplus^{\mathrm{id}} E_\lambda$, $\lambda \in \mathbb{D}$, where $U_\lambda = \overline{\lambda} 1_{E_\lambda}$. All $\mathcal{L}_\lambda$ are subsimple algebras of class (**II**). Set $\mathcal{M} = \oplus_{\lambda\in\mathbb{D}} \mathcal{L}_\lambda$. Denote by P_λ the orthogonal projections in l^2 onto subspaces E_λ. The map θ: $\mathcal{L} \to \mathcal{M}$ defined by the rule

$$\theta(\alpha U \oplus^{\mathrm{id}} x) = \oplus_{\lambda\in\mathbb{D}}(\alpha U_\lambda \oplus^{\mathrm{id}} P_\lambda x) \in \mathcal{M}$$

is a homomorphism, because $P_\lambda U = \overline{\lambda} P_\lambda$ for each $\lambda \in \mathbb{D}$, since $U^* P_\lambda = \lambda P_\lambda$. Furthermore θ is injective. Indeed, if $\theta(\alpha U \oplus^{\mathrm{id}} x) = 0$ then $\alpha = 0$ and $P_\lambda x = 0$ for all $\lambda \in \mathbb{D}$. Hence $(x, e_\lambda) = \sum_n (x, e_n)\lambda^n = 0$, for all $\lambda \in \mathbb{D}$. Thus all $(x, e_n) = 0$, so that $x = 0$. Finally, the projection of the image $\theta(\mathcal{L})$ on each component $\mathcal{L}_\lambda$ clearly coincides with $\mathcal{L}_\lambda$. Therefore $\theta(\mathcal{L})$ is a subdirect product of the Lie algebras $\mathcal{L}_\lambda$.

Theorem 9.9 gives a fairly transparent description of the class of finite-dimensional Frattini-free algebras as direct sums of simple "model" examples; we will return to this subject in the next subsection. Note that in general the subdirect sums can be indecomposable even in the commutative case (take any Banach space of bounded analytic functions).

We shall now consider the structure of some special types of Lie algebras from $\mathbf{Sem}(P_{\mathfrak{S}^{\max}})$.

Theorem 9.10. (i) *If a Lie algebra $\mathcal{L} \in \mathbf{Sem}(P_{\mathfrak{S}^{\max}})$ is solvable then $\mathcal{L}_{[2]} = \{0\}$.*

(ii) *Let $\mathcal{L} \in \mathbf{Sem}(P_{\mathfrak{S}^{\max}})$. Then $\mathcal{L}$ is nilpotent if and only if $\mathcal{L}$ is commutative.*

(iii) *Any solvable Lie algebra in $\mathbf{Sem}(P_{\mathfrak{J}^{\max}})$ is commutative.*

Proof. Let $\mathcal{L} \in \mathbf{Sem}(P_{\mathfrak{S}^{\max}})$. By Proposition 9.6(ii), for each $J \in \mathfrak{J}^{\mathrm{sm}}_{\mathcal{L}}$, either $J \in \mathfrak{S}^{\max}_{\mathcal{L}}$ in which case $\dim(\mathcal{L}/J) = 1$ by (9.1), or there is $\mathcal{M}^J \in \mathfrak{S}^{\max}_{\mathcal{L}}$ such that $\mathcal{M}^J/J$ is a maximal Lie subalgebra of $\mathcal{L}/J$ and it contains no non-zero Lie ideals of $\mathcal{L}/J$.

(i) If $\mathcal{L}$ is solvable, $\mathcal{L}/J$ are solvable for all $J \in \mathfrak{J}^{\mathrm{sm}}_{\mathcal{L}}$. By Corollary 9.5, we have $\dim(\mathcal{L}/J) \le 2$, so that $(\mathcal{L}/J)_{[2]} = \{0\}$. Hence $\mathcal{L}_{[2]} \subseteq J$ for all $J \in \mathfrak{J}^{\mathrm{sm}}_{\mathcal{L}}$. Therefore, by Proposition 9.7, $\mathcal{L}_{[2]} \subseteq \cap_{J\in\mathfrak{J}^{\mathrm{sm}}_{\mathcal{L}}} J = P_{\mathfrak{J}^{\mathrm{sm}}}(\mathcal{L}) = P_{\mathfrak{S}^{\max}}(\mathcal{L}) = \{0\}$.

(ii) If $\mathcal{L}$ is nilpotent then $\mathcal{L}/J$ is nilpotent for each $J \in \mathfrak{J}^{\mathrm{sm}}_{\mathcal{L}}$. By Corollary 9.5, $\dim(\mathcal{L}/J) = 1$. Hence $\mathcal{L}^{[1]} \subseteq J$ for all $J \in \mathfrak{J}^{\mathrm{sm}}_{\mathcal{L}}$. Thus, by Proposition 9.7,

$$\mathcal{L}^{[1]} \subseteq \bigcap_{J\in\mathfrak{J}^{\mathrm{sm}}_{\mathcal{L}}} J = P_{\mathfrak{J}^{\mathrm{sm}}}(\mathcal{L}) = P_{\mathfrak{S}^{\max}}(\mathcal{L}) = \{0\}.$$

(iii) If $\mathcal{L} \in \mathbf{Sem}(P_{\mathfrak{J}^{\max}})$ is solvable then, by Corollary 7.7(ii), $\{0\} = P_{\mathfrak{J}^{\max}}(\mathcal{L}) = \mathcal{L}_{[1]}$. $\square$

The condition $\mathcal{L}_{[2]} = \{0\}$ is not sufficient for a Banach Lie algebra $\mathcal{L}$ to belong to $\mathbf{Sem}(P_{\mathfrak{S}^{\max}})$. Indeed, if $\mathcal{L}$ is the Heisenberg 3-dimensional Lie algebra, as in Example 7.20(ii), then $\mathcal{L}_{[2]} = \{0\}$ and $\mathcal{L} \notin \mathbf{Sem}(P_{\mathfrak{S}^{\max}})$.

The following corollary shows that for Frattini-free algebras there is a natural analogue of the classical solvable radical.

Corollary 9.11. *Each $\mathcal{L} \in \mathbf{Sem}(P_{\mathfrak{S}^{\max}})$ has the largest solvable (commutative) Lie ideal — the closed solvable (commutative) Lie ideal that contains all solvable (commutative) Lie ideals of $\mathcal{L}$.*

Proof. Let $\mathcal{L} \in \mathbf{Sem}(P_{\mathfrak{S}^{\max}})$. The set $\mathcal{E}$ of all closed solvable Lie ideals of $\mathcal{L}$ is partially ordered by inclusion. If I is a closed ideal of $\mathcal{L}$ then $I \in \mathbf{Sem}(P_{\mathfrak{S}^{\max}})$ by Lemma 3.6(iii). Therefore $I_{[2]} = \{0\}$ for each $I \in \mathcal{E}$, by Theorem 9.10(i). Hence if $\{I_\lambda\}_{\lambda\in\Lambda}$ is a linearly ordered subset of $\mathcal{E}$, then the ideal $I = \cup_{\lambda\in\Lambda} I_\lambda$ satisfies $I_{[2]} = \{0\}$. Therefore its closure $\overline{I}$ also satisfies $\overline{I}_{[2]} = \{0\}$, so that $\overline{I} \in \mathcal{E}$. By Zorn's Lemma, $\mathcal{E}$ has a maximal element J.

Let $I \in \mathcal{E}$. Then $I + J$ is a Lie ideal of $\mathcal{L}$, $J \subseteq I + J$ and $(I + J)_{[1]} \subseteq I_{[1]} + J$. Hence $(I + J)_{[2]} \subseteq (I_{[1]} + J)_{[1]} \subseteq I_{[2]} + J = J$. Thus $(I + J)_{[4]} \subseteq J_{[2]} = \{0\}$, so that $\overline{I + J} \in \mathcal{E}$ and $J \subseteq \overline{I + J}$. As J is maximal, $I \subseteq J$, so that J contains all solvable Lie ideals of $\mathcal{L}$.

As above, the set $\mathcal{E}_{\mathrm{c}}$ of all closed commutative Lie ideals of $\mathcal{L}$ is partially ordered by inclusion and contains a maximal element K. Let $I \in \mathcal{E}_{\mathrm{c}}$. Then $I+K$ is a Lie ideal of $\mathcal{L}$, $K \subseteq I+K$ and (see 6.2)) $(I+K)^{[2]} \subseteq I \cap K$. Hence $(I+K)^{[3]} \subseteq [I+K, I\cap K] = \{0\}$. Hence $\overline{I+K}^{[3]} = \{0\}$, so that $\overline{I+K}$ is nilpotent. By Theorem 9.10(ii), $\overline{I+K}$ is commutative. Thus $\overline{I+K} \in \mathcal{E}_{\mathrm{c}}$ and $K \subseteq \overline{I+K}$. As K is maximal, $I \subseteq K$, so that K contains all commutative Lie ideals of $\mathcal{L}$. $\square$

9.3. **Frattini- and Jacobson-free finite-dimensional Lie algebras.** The general description of $\mathrm{P}_{\mathfrak{J}^{\max}}$-semisimple and $\mathrm{P}_{\mathfrak{S}^{\max}}$-semisimple Banach Lie algebras in terms of semidirect products of subsimple algebras (Theorem 9.9) enables one to obtain sufficiently simple "models" for such algebras in the finite-dimensional case.

We say that a Lie algebra L of operators on a finite-dimensional linear space X is *decomposable* if X decomposes into the direct sum of minimal subspaces invariant for L: $X = X_1 \dotplus ... \dotplus X_n$ where all X_k are invariant for L and the restriction of L to each X_k is irreducible. A representation of a Lie algebra will be called decomposable if its image is decomposable.

Lemma 9.12. *Let π be a decomposable representation of a Lie algebra L on a finite-dimensional space X and let $\mathcal{L} = L \oplus^{\pi} X$ (see (3.9)). Then $P_{\mathfrak{S}^{\max}}(\mathcal{L}) \subseteq (\ker \pi \cap P_{\mathfrak{S}^{\max}}(L)) \oplus^{\pi} \{0\}$.*

Proof. We have $X = X_1 \dotplus ... \dotplus X_n$ where all X_k are invariant for π and all restrictions $\pi|_{X_k}$ are irreducible. Then all $M_k = L \oplus^{\pi} (X - X_k)$ are maximal Lie subalgebras of $\mathcal{L}$, so that the Lie ideal $P_{\mathfrak{S}^{\max}}(\mathcal{L}) \subseteq \cap_k M_k = L \oplus^{\pi} \{0\}$. Let $(a,0) \in P_{\mathfrak{S}^{\max}}(\mathcal{L})$. If $a \notin \ker \pi$ then $\pi(a)x \neq 0$ for some $x \in X$. Hence $[(a,0),(0,x)] = (0, \pi(a)x) \in P_{\mathfrak{S}^{\max}}(\mathcal{L})$ – a contradiction. Thus $P_{\mathfrak{S}^{\max}}(\mathcal{L}) \subseteq \ker \pi \oplus^{\pi} \{0\}$. Using Proposition 3.11(i), we conclude the proof. $\square$

Corollary 9.13. *A finite-dimensional Lie algebra $\mathcal{L}$ is Frattini-free if and only if it is isomorphic to the direct sum of Lie algebras of the following types*:

(i) *one-dimensional algebras*;

(ii) *simple Lie algebras*;

(iii) *Lie algebras $L \oplus^{\mathrm{id}} X$, where L is a decomposable Lie algebra of operators on a linear space X.*

Proof. The subsimple Lie algebras in (i), (ii) are Frattini-free by Lemma 9.2. The Lie algebras $\mathcal{L} = L \oplus^{\mathrm{id}} X$ in (iii) are also Frattini-free: by Lemma 9.12, $P_{\mathfrak{S}^{\max}}(\mathcal{L}) = \{0\}$ as $\ker(\mathrm{id}) = \{0\}$.

Conversely, let $\mathcal{L}$ be a Frattini-free Lie algebra. If it decomposes in the direct sum of Lie ideals then, as the preradical $P_{\mathfrak{S}^{\max}}$ is balanced, each of them is Frattini-free. Hence we will assume that $\mathcal{L}$ does not decompose in the direct sum of Lie ideals.

Theorem 9.9 implies that $\mathcal{L}$ can be identified with a subdirect product of some set Λ of subsimple algebras $\{\mathcal{L}_\lambda\}_{\lambda \in \Lambda}$. For each $\lambda \in \Lambda$, let ψ_λ be the homomorphism from $\oplus_\Lambda \mathcal{L}_\lambda$ onto $\mathcal{L}_\lambda$ (see (9.2)). We may assume that Λ is finite. Indeed, for each $\lambda \in \Lambda$, $N_\lambda := \ker \psi_\lambda$ is a Lie ideal of $\mathcal{L}$ and $\cap_{\lambda \in \Lambda} N_\lambda = \{0\}$. As $\dim \mathcal{L} < \infty$, there is a finite subfamily $\lambda_1, ..., \lambda_n$ of Λ with

$$\cap_{i=1}^{n} N_{\lambda_i} = \{0\}. \tag{9.3}$$

Choose the least possible n in (9.3). It follows that $\mathcal{L}$ is isomorphic to a subdirect product of the direct product $\mathcal{M} = \oplus_{i=1}^{n} \mathcal{L}_i$, where $\mathcal{L}_i = \mathcal{L}_{\lambda_i}$. Set $N_i = N_{\lambda_i}$ and $\psi_i = \psi_{\lambda_i}$.

Using the description of subsimple algebras in Theorem 9.4, we may assume that each $\mathcal{L}_i$ is either one-dimensional or a simple Lie algebra or isomorphic to $L_i \oplus^{\mathrm{id}} X_i$, where L_i is an irreducible Lie algebra of operators on a linear finite-dimensional space X_i.

If $n = 1$, the theorem is proved. Let $n > 1$. Then $\cap_{i=2}^{n} N_i$ is a Lie ideal of $\mathcal{L}$. As $\mathcal{L}_1 = \psi_1(\mathcal{L})$, we have that $J_1 = \psi_1(\cap_{i=2}^{n} N_i)$ is a Lie ideal of $\mathcal{L}_1$. If $J_1 = \{0\}$ then $\cap_{i=2}^{n} N_i \subseteq N_1 = \ker \psi_1$, so that $\cap_{i=2}^{n} N_i = \{0\}$ which contradicts the fact that n is the least in (9.3).

If $J_1 = \mathcal{L}_1$ then, for each $x \in \mathcal{L}$, there is $y_x \in \cap_{i=2}^{n} N_i$ such that $\psi_1(x) = \psi_1(y_x)$. Hence $x = y_x + (x - y_x)$ and $x - y_x \in \ker \psi_1 = N_1$. As $(\cap_{i=2}^{n} N_i) \cap N_1 = \{0\}$ by (9.3), we have that $\mathcal{L} = (\cap_{i=2}^{n} N_i) \oplus N_1$ is the direct sum of its Lie ideals. This contradicts our assumption. Thus $\{0\} \neq J_1 \neq \mathcal{L}_1$, so that $\mathcal{L}_1 = L_1 \oplus^{\mathrm{id}} X_1$. As the Lie ideal $\{0\} \oplus^{\mathrm{id}} X_1$ is contained in each Lie ideal of $\mathcal{L}_1$, it is contained in J_1 and, hence, in $\mathcal{L}$.

The similar argument shows that simple and one-dimensional summands are absent in $\mathcal{M}$ and each $\mathcal{L}_i = L_i \oplus^{\mathrm{id}} X_i$. Moreover, $\mathcal{L}$ contains the Lie ideal $\{0\} \oplus^{\mathrm{id}} X$, where $X = \sum_{k=1}^{n} \dotplus X_i$.

Set $M = \oplus_{i=1}^{n} L_i$. Clearly, M can be considered as a Lie algebra of operators on X, preserving each X_i and irreducible on it, and $\mathcal{M} = M \oplus^{\mathrm{id}} X$. As $\mathcal{L} \subseteq \mathcal{M}$ and contains $\{0\} \oplus^{\mathrm{id}} X$, there is a Lie subalgebra L of M such that $\mathcal{L} = L \oplus^{\mathrm{id}} X$. As $\mathcal{L}$ is a subdirect product, $\psi_i(\mathcal{L}) = \mathcal{L}_i = L_i \oplus^{\mathrm{id}} X_i$ for each i. As $\psi_i(\{0\} \oplus^{\mathrm{id}} X) = \{0\} \oplus^{\mathrm{id}} X_i$, we have $\psi_i(L \oplus^{\mathrm{id}} \{0\}) = L_i \oplus^{\mathrm{id}} \{0\}$. Thus $L|_{X_i} \approx L_i$ is irreducible on X_i, so that L is decomposable. $\square$

One can easily deduce from Corollary 9.13 the characterization of finite-dimensional Frattini-free Lie algebras obtained by Stitzinger [S] and Towers [T]. For this we will use the following well known result (see for example [Ch, Proposition 4.4.2.3]).

Lemma 9.14. *Let L be a decomposable Lie algebra of operators on a finite-dimensional space $X = X_1 \dotplus \ldots \dotplus X_n$, where all X_i are irreducible components. Let Z_L be the centre of L. Then*
(i) *$a|_{X_i} = \lambda_i(a)\mathbf{1}_{X_i}$, for all $a \in Z_L$ and i, where λ_i are linear functionals on Z_L;*
(ii) *$[L, L]$ is semisimple and $L = [L, L] \oplus Z_L$.*

In fact, for a finite-dimensional Lie algebra L the conditions $L = [L, L] \oplus Z_L$ and $[L, L]$ is semisimple in (ii) are equivalent ([Ch, Proposition 4.4.2.1]); the Lie algebras satisfying these conditions are called *reductive.*

Corollary 9.15. [S, T]*A finite-dimensional Lie algebra $\mathcal{L}$ is Frattini-free if and only if it is the direct space sum $\mathcal{L} = C \dotplus S \dotplus J$, where J is a commutative Lie ideal of $\mathcal{L}$, C is a commutative Lie subalgebra of $\mathcal{L}$ whose adjoint representation on J is decomposable and S is a semisimple Lie subalgebra of $\mathcal{L}$ such that $[C, S] = \{0\}$.*

Proof. Let $\mathcal{L}$ be Frattini-free. Applying Corollary 9.13, it suffices to obtain the needed decomposition for each direct summand of $\mathcal{L}$. For summands of type (i) and (ii) this is evident. For $\mathcal{L} = L \oplus^{\mathrm{id}} X$, where L is decomposable, set $J = \{0\} \oplus^{\mathrm{id}} X$, $S = [L, L] \oplus^{\mathrm{id}} \{0\}$, $C = Z_L \oplus^{\mathrm{id}} \{0\}$ and apply Lemma 9.14.

Conversely, let $\mathcal{L} = C \dotplus S \dotplus J$ and J, C, S have the properties listed above. Then the Lie algebra $L = C \oplus S$ is reductive and $C = Z_L$. Let $\pi = \mathrm{ad}|_J$ be the adjoint representation of L on J. By our assumptions, the restriction of π to Z_L is decomposable. It follows that π is decomposable (see [Ch, Corollary 4.4.1.2]). As L is the direct sum of a semisimple and commutative Lie ideals, we have $P_{\mathfrak{S}^{\max}}(L) = \{0\}$. Hence, by Lemma 9.12, $P_{\mathfrak{S}^{\max}}(\mathcal{L}) = \{0\}$. $\square$

Recall that $\mathcal{L} \in \mathfrak{L}$ is Jacobson-free if $P_{\mathfrak{J}^{\max}}(\mathcal{L}) = \{0\}$. Similar, but simpler arguments give us the description of Jacobson-free algebras (for a different proof see the end of the paper).

Corollary 9.16. *A finite-dimensional Lie algebra $\mathcal{L}$ is Jacobson-free if and only if $\mathcal{L}$ is the direct sum of a semisimple and a commutative Lie algebras.*

9.4. **Frattini and Jacobson indices of finite-dimensional Lie algebras.** In this section we study the class $\mathfrak{L}^{\mathrm{f}}$ of complex finite-dimensional Lie algebras. As $\{0\}$ is a Lie ideal of finite codimension in each $\mathcal{L} \in \mathfrak{L}^{\mathrm{f}}$, we have $\mathcal{F}(\mathcal{L}) = \{0\}$ and $\mathfrak{L}^{\mathrm{f}} \subseteq \mathbf{Sem}(P_{\mathfrak{J}})$.

The Lie ideal $P_{\mathfrak{S}^{\max}}(\mathcal{L})$ is called the *Frattini ideal* and $P_{\mathfrak{J}^{\max}}(\mathcal{L})$ the *Jacobson ideal* of $\mathcal{L}$ (in [M] it was called the *Jacobson radical*). By Theorem 7.8, $P_{\mathfrak{S}^{\max}}(\mathcal{L}) \subseteq P_{\mathfrak{J}^{\max}}(\mathcal{L})$. The ordinal numbers $r^{\circ}_{P_{\mathfrak{S}^{\max}}}(\mathcal{L})$ and $r^{\circ}_{P_{\mathfrak{J}^{\max}}}(\mathcal{L})$ (see (4.6)) belong to $\mathbb{N}$ and satisfy

$$\{0\} = \mathcal{F}(\mathcal{L}) = P^{\alpha}_{\mathfrak{S}^{\max}}(\mathcal{L}) = P^{\beta}_{\mathfrak{J}^{\max}}(\mathcal{L}), \text{ where } \alpha = r^{\circ}_{P_{\mathfrak{S}^{\max}}}(\mathcal{L}), \beta = r^{\circ}_{P_{\mathfrak{J}^{\max}}}(\mathcal{L}).$$

They are called, respectively, the *Frattini* (see [M, Definitions 4]) and *Jacobson indices* of $\mathcal{L}$. By Theorem 7.8, $r^{\circ}_{P_{\mathfrak{S}^{\max}}}(\mathcal{L}) \le r^{\circ}_{P_{\mathfrak{J}^{\max}}}(\mathcal{L}) < \infty$.

Denote by $\mathcal{N}_{\mathcal{L}}$ the nil-radical of $\mathcal{L}$ — the maximal nilpotent ideal of $\mathcal{L}$. Combining this with results of [M, p. 420 and 422] and [J, Theorem II.7.13], yields

$$P_{\mathfrak{S}^{\max}}(\mathcal{L}) \subseteq P_{\mathfrak{J}^{\max}}(\mathcal{L}) = \mathcal{K}_{\mathcal{L}} \subseteq \mathcal{N}_{\mathcal{L}} \subseteq \mathrm{rad}(\mathcal{L}) \text{ where } \mathcal{K}_{\mathcal{L}} = [\mathcal{L}, \mathrm{rad}(\mathcal{L})]. \tag{9.4}$$

For a solvable Lie algebra $\mathcal{L}$, the *solvability index* $i_s(\mathcal{L})$ is the least n such that $\mathcal{L}_{[n]} = 0$. Marshall [M, p. 421] proved that $r^{\circ}_{P_{\mathfrak{S}^{\max}}}(\mathcal{L}) \le i_s(\mathcal{N}_{\mathcal{L}}) + 1$. Below we refine this result.

Proposition 9.17. (i) *If $\mathcal{L}$ is nilpotent,* $r^{\circ}_{P_{\mathfrak{S}^{\max}}}(\mathcal{L}) = r^{\circ}_{P_{\mathfrak{J}^{\max}}}(\mathcal{L}) = i_s(\mathcal{L})$.

(ii) *If $\mathcal{L}$ is a finite-dimensional complex Lie algebra, then*

$$i_s(\mathcal{N}_{\mathcal{L}}) \le r^{\circ}_{P_{\mathfrak{S}^{\max}}}(\mathcal{L}) \le r^{\circ}_{P_{\mathfrak{J}^{\max}}}(\mathcal{L}) = i_s(\mathcal{K}_{\mathcal{L}}) + 1 \le i_s(\mathcal{N}_{\mathcal{L}}) + 1, \tag{9.5}$$

so that $1 \le r^{\circ}_{P_{\mathfrak{S}^{\max}}}(\mathcal{L}) \le r^{\circ}_{P_{\mathfrak{J}^{\max}}}(\mathcal{L}) \le r^{\circ}_{P_{\mathfrak{S}^{\max}}}(\mathcal{L}) + 1$.

Proof. (i) If $\mathcal{L}$ is nilpotent then (see [M, p. 420]) every maximal Lie subalgebra is a Lie ideal, so that $P_{\mathfrak{S}^{\max}}(\mathcal{L}) = P_{\mathfrak{J}^{\max}}(\mathcal{L})$. Hence, by (9.4), $P_{\mathfrak{S}^{\max}}(\mathcal{L}) = P_{\mathfrak{J}^{\max}}(\mathcal{L}) = \mathcal{K}_{\mathcal{L}} = \mathcal{L}_{[1]}$. Thus

$$P^k_{\mathfrak{S}^{\max}}(\mathcal{L}) = P^k_{\mathfrak{J}^{\max}}(\mathcal{L}) = \mathcal{L}_{[k]} \text{ for each } k, \tag{9.6}$$

so that $r^{\circ}_{P_{\mathfrak{S}^{\max}}}(\mathcal{L}) = r^{\circ}_{P_{\mathfrak{J}^{\max}}}(\mathcal{L}) = i_s(\mathcal{L})$.

(ii) By (9.4), $P_{\mathfrak{S}^{\max}}(\mathcal{L})$ and $P_{\mathfrak{J}^{\max}}(\mathcal{L})$ are nilpotent for each $\mathcal{L} \in \mathfrak{L}^{\mathrm{f}}$. Hence, by (9.6),

$$P^k_{\mathfrak{S}^{\max}}(\mathcal{L}) = P^{k-1}_{\mathfrak{S}^{\max}}(P_{\mathfrak{S}^{\max}}(\mathcal{L})) = P_{\mathfrak{S}^{\max}}(\mathcal{L})_{[k-1]},$$
$$P^k_{\mathfrak{J}^{\max}}(\mathcal{L}) = P^{k-1}_{\mathfrak{J}^{\max}}(P_{\mathfrak{J}^{\max}}(\mathcal{L})) = P_{\mathfrak{J}^{\max}}(\mathcal{L})_{[k-1]}.$$

Let R be $P_{\mathfrak{S}^{\max}}$ or $P_{\mathfrak{J}^{\max}}$. By (4.6), $r^{\circ}_R(\mathcal{L})$ is the least n such that $R^n(\mathcal{L}) = \{0\}$. Thus

$$r^{\circ}_{P_{\mathfrak{S}^{\max}}}(\mathcal{L}) = i_s(P_{\mathfrak{S}^{\max}}(\mathcal{L})) + 1 \text{ and } r^{\circ}_{P_{\mathfrak{J}^{\max}}}(\mathcal{L}) = i_s(P_{\mathfrak{J}^{\max}}(\mathcal{L})) + 1. \tag{9.7}$$

Hence, by (9.4) and (9.7),

$$r^{\circ}_{P_{\mathfrak{J}^{\max}}}(\mathcal{L}) = i_s(P_{\mathfrak{J}^{\max}}(\mathcal{L})) + 1 = i_s(\mathcal{K}_{\mathcal{L}}) + 1 \le i_s(\mathcal{N}_{\mathcal{L}}) + 1. \tag{9.8}$$

As $P_{\mathfrak{S}^{\max}}$ is balanced and $\mathcal{N}_{\mathcal{L}}$ is nilpotent, we obtain $(\mathcal{N}_{\mathcal{L}})_{[1]} = P_{\mathfrak{S}^{\max}}(\mathcal{N}_{\mathcal{L}}) \subseteq P_{\mathfrak{S}^{\max}}(\mathcal{L})$ from (9.6). Hence $(\mathcal{N}_{\mathcal{L}})_{[k+1]} \subseteq P_{\mathfrak{S}^{\max}}(\mathcal{L})_{[k]}$, so that $i_s(\mathcal{N}_{\mathcal{L}}) \le i_s(P_{\mathfrak{S}^{\max}}(\mathcal{L})) + 1$. Combining this with (9.7) and (9.8) and taking into account that $r^{\circ}_{P_{\mathfrak{S}^{\max}}}(\mathcal{L}) \le r^{\circ}_{P_{\mathfrak{J}^{\max}}}(\mathcal{L})$, we have (9.5). ☐

For ordinals α and β with $\alpha \le \beta$ set

$$\mathfrak{L}_{(\alpha,\beta)} = \{\mathcal{L} \in \mathbf{Sem}(\mathcal{F}) : r^{\circ}_{P_{\mathfrak{S}^{\max}}}(\mathcal{L}) = \alpha,\ r^{\circ}_{P_{\mathfrak{J}^{\max}}}(\mathcal{L}) = \beta\}.$$

From Proposition 9.17(ii) it follows that $\mathfrak{L}^{\mathrm{f}}$ can be partitioned into three following classes:

$$\mathrm{C}_1 = \{\mathcal{L} \in \mathfrak{L}^{\mathrm{f}} : r^{\circ}_{P_{\mathfrak{S}^{\max}}}(\mathcal{L}) = r^{\circ}_{P_{\mathfrak{J}^{\max}}}(\mathcal{L}) = i_s(\mathcal{K}_{\mathcal{L}}) + 1 = i_s(\mathcal{N}_{\mathcal{L}}) + 1\};$$
$$\mathrm{C}_2 = \{\mathcal{L} \in \mathfrak{L}^{\mathrm{f}} : r^{\circ}_{P_{\mathfrak{S}^{\max}}}(\mathcal{L}) = r^{\circ}_{P_{\mathfrak{J}^{\max}}}(\mathcal{L}) = i_s(\mathcal{K}_{\mathcal{L}}) + 1 = i_s(\mathcal{N}_{\mathcal{L}})\};$$
$$\mathrm{C}_3 = \{\mathcal{L} \in \mathfrak{L}^{\mathrm{f}} : r^{\circ}_{P_{\mathfrak{S}^{\max}}}(\mathcal{L}) + 1 = r^{\circ}_{P_{\mathfrak{J}^{\max}}}(\mathcal{L}) = i_s(\mathcal{K}_{\mathcal{L}}) + 1 = i_s(\mathcal{N}_{\mathcal{L}}) + 1\},$$

Hence

$$\mathrm{C}_1 \cup \mathrm{C}_2 \cup \mathrm{C}_3 = \mathfrak{L}^{\mathrm{f}},\ \mathrm{C}_1 \cup \mathrm{C}_2 = \cup_{n\ge1}(\mathfrak{L}_{(n,n)} \cap \mathfrak{L}^{\mathrm{f}}) \text{ and } \mathrm{C}_3 = \cup_{n\ge1}(\mathfrak{L}_{(n,n+1)} \cap \mathfrak{L}^{\mathrm{f}}).$$

It follows from Proposition 9.17(i) and the above formulae that

$$\{\mathcal{L} \in \mathfrak{L}^{\mathrm{f}} : \mathcal{L} \text{ is nilpotent}\} \subseteq C_2, \text{ so that } \mathfrak{L}_{(n,n)} \ne \varnothing \text{ for all } n \ge 1;$$

$\mathrm{C}_1 \cap \mathfrak{L}_{(1,1)} = \{\mathcal{L} \in \mathfrak{L}^{\mathrm{f}}\colon \mathcal{L} \text{ is semisimple}\}$;

$\mathrm{C}_2 \cap \mathfrak{L}_{(1,1)} = \{\mathcal{L} \in \mathfrak{L}^{\mathrm{f}}\colon \mathcal{L} = N_{\mathcal{L}} \oplus \mathrm{rad}(\mathcal{L}),\ N_{\mathcal{L}} \text{ is semisimple, } \mathrm{rad}(\mathcal{L}) \ne \{0\} \text{ is commutative}\}$.

Consider the solvable Lie algebra $\mathcal{L}$ of all upper triangular $n \times n$ matrices. Then $\mathcal{K}_{\mathcal{L}} = \mathcal{L}_{[1]} = \mathcal{N}_{\mathcal{L}}$ is the nilpotent Lie subalgebra of $\mathcal{L}$ of all matrices with zero on the diagonal. The Lie subalgebras $\mathcal{L}_{kk} = \{a = (a_{ij}) \in \mathcal{L}\colon a_{kk} = 0\}$, $1 \le k \le n$, and $\mathcal{L}_{k,k+1} = \{a = (a_{ij}) \in \mathcal{L}\colon a_{k,k+1} = 0\}$, $1 \le k \le n-1$, have codimension 1 in $\mathcal{L}$, so that they are maximal. Hence

$$P_{\mathfrak{S}^{\max}}(\mathcal{L}) \subseteq (\cap_k \mathcal{L}_{kk}) \cap (\cap_k \mathcal{L}_{k,k+1}) = \mathcal{L}_{[2]}.$$

Therefore

$$r^{\circ}_{P_{\mathfrak{S}^{\max}}}(\mathcal{L}) \overset{(9.7)}{=} i_s(P_{\mathfrak{S}^{\max}}(\mathcal{L})) + 1 \le i_s(\mathcal{L}_{[2]}) + 1 \text{ and } r^{\circ}_{P_{\mathfrak{J}^{\max}}}(\mathcal{L}) \overset{(9.8)}{=} i_s(\mathcal{K}_{\mathcal{L}}) + 1 = i_s(\mathcal{L}_{[1]}) + 1,$$

so that $r^{\circ}_{P_{\mathfrak{S}^{\max}}}(\mathcal{L}) + 1 \le r^{\circ}_{P_{\mathfrak{J}^{\max}}}(\mathcal{L})$. Thus, by Proposition 9.17,

$$r^{\circ}_{P_{\mathfrak{S}^{\max}}}(\mathcal{L}) + 1 = r^{\circ}_{P_{\mathfrak{J}^{\max}}}(\mathcal{L}) = i_s(\mathcal{K}_{\mathcal{L}}) + 1 = i_s(\mathcal{L}_{[1]}) + 1 = n.$$

Then $\mathcal{L} \in \mathfrak{L}_{(n,n+1)}$. Combining this and Proposition 9.17 yields

Corollary 9.18. *$\mathfrak{L}^{\mathrm{f}} \subseteq \cup_n \left(\mathfrak{L}_{(n,n)} \cup \mathfrak{L}_{(n,n+1)}\right)$ and all classes $\mathfrak{L}_{(n,n)}$ and $\mathfrak{L}_{(n,n+1)}$ contain finite-dimensional Lie algebras.*

Proposition 9.17 also gives us a proof of Corollary 9.16.

Proof of Corollary 9.16. Let $\mathcal{L} \in \mathbf{Sem}(P_{\mathfrak{J}^{\max}}) \cap \mathfrak{L}^{\mathrm{f}}$. Then $r^{\circ}_{P_{\mathfrak{J}^{\max}}}(\mathcal{L}) = 1$ and, by (9.5), $i_s(\mathcal{K}_{\mathcal{L}}) = 0$. Hence $\mathcal{K}_{\mathcal{L}} = [\mathcal{L}, \mathrm{rad}\,(\mathcal{L})] = \{0\}$, so that $\mathrm{rad}(\mathcal{L})$ is the centre $Z_{\mathcal{L}}$ of $\mathcal{L}$. As $\mathcal{L} = N_{\mathcal{L}} \oplus^{\mathrm{ad}} \mathrm{rad}(\mathcal{L})$ is the semidirect product of a semisimple Lie algebra $N_{\mathcal{L}}$ and $\mathrm{rad}(\mathcal{L})$, we have that $\mathcal{L} = N_{\mathcal{L}} \oplus Z_{\mathcal{L}}$.

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